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LASER BEAM PROPAGATION IN PARTICULATE MEDIA

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(6) LASER BEAM PROPAGATION IN PARTICULATE MEDIA

by

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RESUME

Nous présentons, dans ce rapport, une théorie de diffusion multiple de la lumière pour des petits angles, dérivée de l'interaction des aérosols sur le rayon laser. Le calcul de la luminance énergétique diffusée peut être effectué à n'importe lequel degré de multiplicité de diffusion et on y donne des résultats exacts. Nous comparons les résultats numériques obtenus en suivant cette théorie avec ceux atteints par le biais d'une théorie approximative en usage. (NC)

ABSTRACT

A theory for small angle multiple scattering of a laser beam by aerosols is presented. Unlike in previous studies, the present formulation yields exact results with which the scattered radiance can be computed to any desired order of multiple scattering. Numerical results based on the exact theory are compared with those derived from an approximate theory currently in use. (U)

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1.0 INTRODUCTION

The performance of optical communication systems using laser sources is often limited by the occurrence of low-visibility meteorological conditions. This is due to the presence of a large number of aerosol particles in the atmosphere. These particles scatter as well as absorb radiant energy from an incident laser beam. As a result, a well collimated beam will disperse more readily in an atmosphere heavily laden with aerosols and the beam intensity will rapidly decrease correspondingly. To evaluate the operational range of a given optical system, it is necessary to describe quantitatively the degradation of optical signals by aerosols.

It is well known that, in a clear atmosphere, the detection of laser illumination of a military target can be achieved only when the detector is situated close to the beam axis. This is because the laser beam used has a small angular divergence. To enhance the off-axis detectability, it has been suggested that aerosol particles may be injected into the atmosphere so that radiant energy can be scattered into a wider angular region.

The purpose of this report is to formulate a theory to calculate the intensity of a laser beam scattered by aerosols. Since we are mainly interested in optically dense media, our starting point is Wentzel's summation method which includes explicitly multiple scatterings of all orders. Assuming the scattering phase function to be strongly peaked in the forward direction, Arnush and Stotts (Refs 1, 2) have recently studied problems similar to the present one. However, to arrive at their analytic results further approximations were made. On the other hand, these additional approximations are not necessary in our treatment. Thus, for aerosols with a highly anisotropic phase function, the present result is exact.

In Section 2, Wentzel's summation method is first discussed and applied to the problem of small angle multiple scattering of a collimated laser beam. It is also shown that similar results can be obtained by starting from the radiative transfer equation. Section 3 contains numerical results based on analytic expressions of Section 2. Comparison with results derived from applying Arnush - Stotts type of approximation (Refs 1, 2) are also made. The main conclusions of this report are contained in Section 4.

This work was performed at DREV between March and November, 1977 under PCN 33A11, Aerosol Studies.

2.0 SMALL ANGLE MULTIPLE SCATTERING OF A COLLIMATED LASER BEAM

The collimated laser beam we consider can be regarded as a monochromatic pencil of light with a small angular divergence. The beam propagates in a medium whose scattering phase function is assumed to be highly anisotropic with a strong peak in the forward direction. This assumption is valid, for example, for the propagation of visible and near infrared radiation in haze and fog. Hence, the majority of the scattering events are characterized by the small scattering angles. In our discussion, we do not consider the effects of polarization and, furthermore, we assume that the optical properties of the medium are not affected by the propagation of the laser beam.

To formulate a theory for small angle multiple scattering of the laser beam, two different approaches will be followed. On the basis of the summation method of Wentzel (Ref. 3), the intensity of the scattered beam can be written as an infinite series, each term of which corresponding to the contribution to the intensity by the scattering of a given order. This infinite series can be summed and thus, numerical evaluation of the intensity is not restricted to the lower order terms.

It is evident that the present problem is a special case of the general question of the solution of the radiative transfer equation. It is well known that even in the situation of a plane parallel medium uniformly irradiated by a source, the solution of the radiative transfer equation is non trivial (Ref. 4). With the additional complication of non uniform irradiation, the problem becomes more difficult and has not yet been solved in the general form. We will show that, with the small angle approximation, the radiative transfer equation can be solved for the multiple scattering of a collimated laser beam. The solution thus obtained is identical to the one found by using Wentzel's method.

2.1 The Small Angle Approximation

The small angle approximation is of central importance for our subsequent discussion. In order to give a more precise statement of the content of this approximation, let us first introduce a few quantities and specify the geometry of the problem.

We choose a coordinate system such that the laser beam has its source at the origin, and the beam axis, along the z-axis (Fig. 1). The intensity $I(\vec{r}, \vec{s})$ is a function of the position vector $\vec{r} = (x, y, z)$ and a unit vector $\vec{s}$ specifying a given direction. The direction $\vec{s}$ may be given in terms of the polar angles (θ, ϕ), or the projected angles (ϕ_x, ϕ_y), (Fig. 2). It is clear that the projected and the polar angles are related by:

$$\tan \phi_x = \tan \theta \cos \phi \tag{1}$$

$$\tan \phi_y = \tan \theta \sin \phi \tag{2}$$

The intensity may be denoted also by $I(x, y, z, \theta, \phi)$ or $I(x, y, z, \phi_x, \phi_y)$. In the small angle approximation, it is assumed that one can replace $\sin \theta$ by θ, and $\cos \theta$ by 1. In addition, all the functions, with

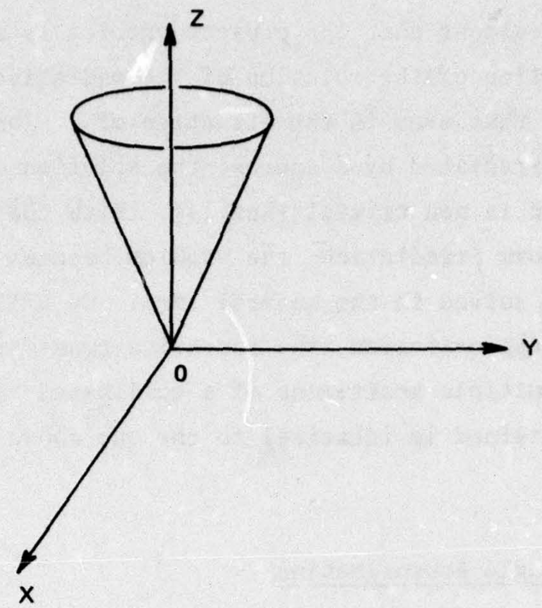

FIGURE 1 - Geometrical configuration

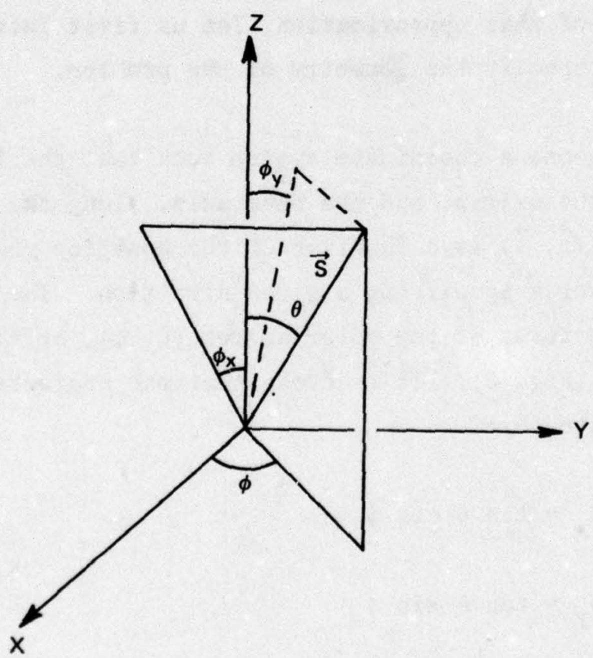

FIGURE 2 - Polar and projected angles

θ, ϕ_x, ϕ_y as arguments, which occur in $I(\vec{r}, \vec{s})$ are assumed to decrease rapidly as θ, $|\phi_x|$, $|\phi_y|$ increase so that, if $g(\vec{r}, \phi_x, \phi_y)$ is such a function, then, the following approximation

$$\int_{-\pi}^{\pi} (2)\ g(\vec{r}, \phi_x, \phi_y)\ d\phi_x\ d\phi_y = \int_{-\infty}^{\infty} (2)\ g(\vec{r}, \phi_x, \phi_y)\ d\phi_x\ d\phi_y \tag{3}$$

is valid. In the above equation, (2) indicates a 2-fold integration.

Consider a photon with initial direction of propagation (ϕ_{x_0}, ϕ_{y_0}) at the origin. It suffers m scatterings without being absorbed in traversing a distance z, and the corresponding scattering angles are $(\bar{\phi}_{x_1}, \bar{\phi}_{y_1})$, $(\bar{\phi}_{x_2}, \bar{\phi}_{y_2})$, . . ., $(\bar{\phi}_{x_m}, \bar{\phi}_{y_m})$; (Fig. 3). On emerging from the plane z, the direction of the photon has (ϕ_x, ϕ_y) as the projection angles

$$\phi_x = \phi_{x_0} + \sum_{j=1}^{m} \bar{\phi}_{x_j} \tag{4}$$

$$\phi_y = \phi_{y_0} + \sum_{j=1}^{m} \bar{\phi}_{y_1} \tag{5}$$

In the small angle approximation, $\tan \phi_x \simeq \phi_y$, $\tan \phi_y \simeq \phi_y$, the lateral coordinates (x, y) of the photon in the plane z are given by

$$x = z\phi_{x_0} + \sum_{j=1}^{m} (z - z_j)\ \bar{\phi}_{x_j} \tag{6}$$

$$y = z\phi_{y_0} + \sum_{j=1}^{m} (z - z_j)\ \bar{\phi}_{y_j} \tag{7}$$

where z_j is the plane in which the j^{th} scattering occurs.

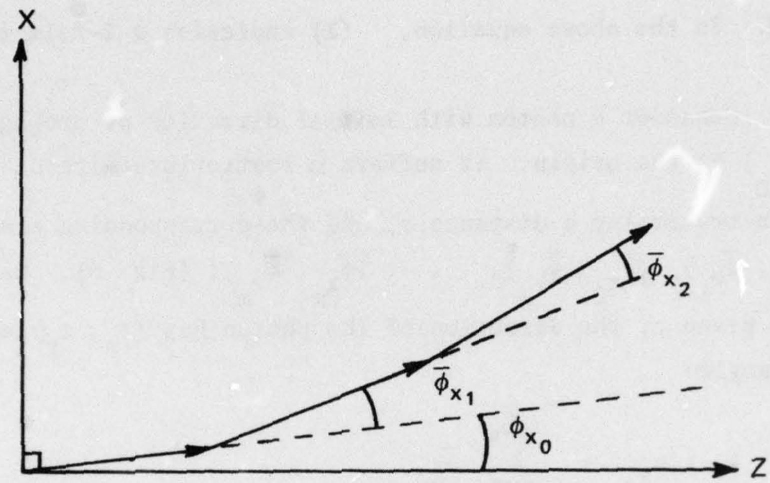

FIGURE 3 - Scattering angles

2.2 Small Angle Multiple Scattering by Wentzel's Method (Ref. 5)

Consider a beam of photons with initial direction of propagation (ϕ_{x_0}, ϕ_{y_0}). We choose a plane of observation z sufficiently close to the source, so that only single scattering needs to be considered. Let $w(\overline{\phi}_x, \overline{\phi}_y, \overline{x}, \overline{y}, z)$ be the single scattering joint distribution function for finding a photon which has undergone a scattering with scattering angles $(\overline{\phi}_x, \overline{\phi}_y)$ and lateral deflections $(\overline{x}, \overline{y})$. If the scattering occurred in the plane z', then,

$$\overline{x} = (z - z')\overline{\phi}_x, \quad y = (z - z')\overline{\phi}_y \tag{8}$$

If $p(\overline{\phi}_x, \overline{\phi}_y, z)$ is the normalized scattering phase function, the joint distribution function $w(\overline{\phi}_x, \overline{\phi}_y, \overline{x}, \overline{y}, z)$ can be expressed as (Ref. 5)

$$w(\overline{\phi}_x, \overline{\phi}_y, \overline{x}, \overline{y}, z) = \frac{\sigma_s}{4\pi} p(\overline{\phi}_x, \overline{\phi}_y, z')\, \delta[\overline{x} - (z - z')\overline{\phi}_x] \;.$$
$$\delta[\overline{y} - (z - z')\overline{\phi}_y] \tag{9}$$

where σ_s is the scattering coefficient.

We remove now the restriction that the plane z is close to the source. The intensity distribution $F(\phi_x, \phi_y, x, y, z)$ corresponding to an initial beam of direction (ϕ_{x_0}, ϕ_{y_0}) can be written as an infinite series

$$F(\phi_x, \phi_y, x, y, z) = \sum_{j=0}^{\infty} F_j(\phi_x, \phi_y, x, y, z) \tag{10}$$

where F_j is the contribution from the j^{th} order scattering. The probability for a photon to pass through a layer of thickness z of the medium without suffering scattering nor absorption is given by

$$\exp\left[- \int_0^z \sigma(z')\, dz' \right]$$

where $\sigma(z)$ is the volume extinction coefficient of the medium in the plane z. It is clear that

$$F_0(\phi_x, \phi_y, x, y, z) = I_0 \exp\left[-\int_0^z \sigma(z')\, dz'\right] \delta(\phi_x - \phi_{x_0}) \quad .$$

$$\delta(\phi_y - \phi_{y_0})\ \delta(x - z\phi_{x_0})\ \delta(y - z\phi_{y_0}) \tag{11}$$

where I_0 is a constant proportional to the photon flux in the incident beam propagating in the direction (ϕ_{x_0}, ϕ_{y_0}). The probability of a photon in the beam, undergoing one scattering with scattering angles $(\bar{\phi}_{x_1}, \bar{\phi}_{y_1})$ and corresponding lateral displacements $(\bar{x}_1, \bar{y}_1)$ is

$$\exp\left[-\int_0^z \sigma(z')\, dz'\right] \int_0^z dz\ w(\bar{\phi}_x, \bar{\phi}_y, \bar{x}, \bar{y}, z') \tag{12}$$

The second factor in (12) gives the probability of one scattering occurring at any point between the interval (0, z), and the first factor gives the probability of no scattering nor absorption occurring at any other point. From (12) we can write

$$F_1(\phi_x, \phi_y, x, y, z) = I_0 \exp\left[-\int_0^z \sigma(z')\, dz'\right] \int_0^z dz'\ w(\bar{\phi}_x, \bar{\phi}_y, \bar{x}, \bar{y}, z')$$

$$= I_0 \exp\left[-\int_0^z \sigma(z')\, dz'\right] \int_0^z dz'\ w(\phi_x - \phi_{x_0}, \phi_y - \phi_{y_0}, x - x_0, y - y_0, z') \tag{13}$$

The probability of having exactly two scatterings in the interval (0,z), with scattering angles and lateral deflections given by $(\bar{\phi}_{x_1}, \bar{\phi}_{y_1})$, $(\bar{\phi}_{x_2}, \bar{\phi}_{y_2})$, $(\bar{x}_1, \bar{y}_1)$ and $(\bar{x}_2, \bar{y}_2)$ respectively, is

$$\exp\left[-\int_0^z \sigma(z')\,dz'\right] \cdot \int_0^z w(\bar\phi_{x_1}, \bar\phi_{y_1}, \bar x_1, \bar y_1, z_1)\,dz_1$$

$$\cdot \int_{z_1}^z w(\bar\phi_{x_2}, \bar\phi_{y_2}, \bar x_2, \bar y_2, z_2)\,dz_2$$

In the above expression, the second factor corresponds to the first scattering event, and the third factor to the second scattering. Hence, the limits of integration for the second scattering extends from z_1, (where the first scattering is supposed to have taken place) to z. As usual, the expression can be written in a more symmetric form (Ref. 6) and integrated over $\bar\phi_{x_1}$, $\bar\phi_{y_1}$, $\bar x_1$, $\bar y_1$ to obtain

$$F_2(\phi_x, \phi_y, x, y, z) = \frac{I_0 \exp\left[-\int_0^z \sigma(z')\,dz'\right]}{2!} \int_{-\infty}^{\infty} (4)$$

$$d\bar\phi_{x_1} d\bar\phi_{y_1} d\bar x_1 d\bar y_1 \cdot \left[\int_0^z dz_1 \; w(\bar\phi_{x_1}, \bar\phi_{y_1}, \bar x_1, \bar y_1, z_1) \cdot \right.$$

$$\left. \int_0^z dz_2 \; w(\bar\phi_{x_2}, \bar\phi_{y_2}, \bar x_2, \bar y_2, z_2)\right] \qquad (14)$$

where $\phi_x = \bar\phi_{x_1} + \bar\phi_{x_2} + \phi_{x_0}$, $\phi_y = \bar\phi_{y_1} + \bar\phi_{y_2} + \phi_{y_0}$, $x = \bar x_1 + \bar x_2 + x_0$ and $y = \bar y_1 + \bar y_2 + \bar y_0$. In order to avoid confusion, we use variables with a bar to indicate scattering quantities (scattering angles and deflections produced by a given scattering). Quantities without a bar, on the other hand, are measured from the coordinate axes. The contribution F_j from j^{th} order scattering to F can be written similar to (14) as

$$F_j(\phi_x, \phi_y, x, y, z) = I_0 \frac{\exp\left[-\int_0^z \sigma(z')\,dz'\right]}{j!} \int_{-\infty}^{\infty} (4)\, d\bar{\phi}_{x_1} d\bar{\phi}_{y_1} d\bar{x}_1 d\bar{y}_1$$

$$\cdot \int_{-\infty}^{\infty} (4)\, d\bar{\phi}_{x_2} d\bar{\phi}_{y_2} d\bar{x}_2 d\bar{y}_2 \; \ldots \int_{-\infty}^{\infty} (4)\, d\bar{\phi}_{x_{j-1}} d\bar{\phi}_{y_{j-1}} d\bar{x}_{j-1} d\bar{y}_{j-1}$$

$$\left[\int_0^z dz_1\, w(\bar{\phi}_{x_1}, \bar{\phi}_{y_1}, \bar{x}_1, \bar{y}_1, z_1) \ldots \int_0^z dz_{j-1}\, w(\bar{\phi}_{x_{j-1}}, \bar{\phi}_{y_{j-1}}, \bar{x}_{j-1},\right.$$

$$\left.\bar{y}_{j-1}, z_{j-1}) \cdot \int_0^z dz\, w(\bar{\phi}_{x_j}, \bar{\phi}_{y_1}, \bar{x}_j, \bar{y}_j, z_j)\right] \tag{15}$$

where $\phi_x = \phi_{x_0} + \sum_{i=1}^{j} \bar{\phi}_{x_i}$, $y = \phi_{y_0} + \sum_{i=1}^{j} \bar{\phi}_{y_i}$

$x = x_0 + \sum_{i=1}^{j} \bar{x}_i$ and $y = y_0 + \sum_{i=1}^{j} \bar{y}_i$

Let $\tilde{F}(\xi_x, \xi_y, \zeta_x, \zeta_y, z)$ be the Fourier transform of $F(\phi_x, \phi_y, x, y, z)$ so that

$$\tilde{F}(\xi_x, \xi_y, \zeta_x, \zeta_y, z) = \int_{-\infty}^{\infty} (4)\, e^{i(\xi_x\phi_x + \xi_y\phi_y + \zeta_x x + \zeta_y y)}$$

$$F(\phi_x, \phi_y, x, y, z)\, d\phi_x\, d\phi_y\, dx\, dy \tag{16}$$

From (10) and the linearity of the Fourier transform, we have

$$\tilde{F}(\xi_x, \xi_y, \zeta_x, \zeta_y, z) = \sum_{n=0}^{\infty} \tilde{F}_n(\xi_x, \xi_y, \zeta_x, \zeta_y, z) \tag{17}$$

and

$$\tilde{F}_n = (\xi_x, \xi_y, \zeta_x, \zeta_y, z) = \int_{-\infty}^{\infty} (4)\ F_n(\phi_x, \phi_y, x, y, z)\ d\phi_x d\phi_y dx dy \tag{18}$$

From Eqs. (11) and (13)

$$\tilde{F}_0(\xi_x, \xi_y, \zeta_x, \zeta_y, z) = I_0 e^{-\Omega_0(z)}\ e^{i(\xi_x\phi_{x_0} + \xi_y\phi_{y_0} + z\zeta_x\phi_{x_0} + z\zeta_y\phi_{y_0})} \tag{19}$$

$$\tilde{F}_1(\xi_x, \xi_y, \zeta_x, \zeta_y, z) = I_0 e^{-\Omega_0(z)}\ e^{i(\xi_x\phi_{x_0} + \xi_y\phi_{y_0} + z\zeta_x\phi_{x_0} + z\zeta_y\phi_{y_0}}$$

$$\cdot \int_0^z \tilde{w}(\xi_x, \xi_y, \zeta_x, \zeta_y, z')\ dz' \tag{20}$$

where

$$\Omega_0(z) = \int_0^z \sigma(z')\ dz', \tag{21}$$

$$\tilde{w}(\xi_x, \xi_y, \zeta_x, \zeta_y, z') = \int_{-\infty}^{\infty} (4) \left[e^{i(\xi_x\bar{\phi}_x + \xi_y\bar{\phi}_y + \zeta_x\bar{x} + \zeta_y\bar{y})}\right] .$$

$$w(\bar{\phi}_x, \bar{\phi}_y, \bar{x}, \bar{y}, z)\ d\bar{\phi}_x d\bar{\phi}_y d\bar{x} d\bar{y} \tag{22}$$

Applying the convolution theorem, one can readily show that if $\tilde{f}(\xi)$, $\tilde{g}(\xi)$ are the Fourier transforms of the functions $f(x)$ and $g(x)$ respectively, then, the Fourier transform of

$$\int_{-\infty}^{\infty} f(y)\ g(x - y - y_0)\ dy$$

is $\tilde{f}(\xi)\ \tilde{g}(\xi)\ e^{i\xi y_0}$.

Consequently,

$$\tilde{F}_n(\xi_x, \xi_y, \zeta_x, \zeta_y, z) = I_0 \frac{e^{-\Omega_0(z)}}{n!} e^{i(\xi_x\phi_{x_0} + \xi_y\phi_{y_0} + \zeta_x x_0 + \zeta_y y_0)}$$

$$\cdot \left[\int_0^z \tilde{w}(\xi_x, \xi_y, \zeta_x, \zeta_y, z')\, dz' \right]^n \tag{23}$$

and

$$\tilde{F}(\xi_x, \xi_y, \zeta_x, \zeta_y, z) = I_0\, e^{i\phi_{x_0}(\xi_x + z\zeta_x) + i\phi_{y_0}(\xi_y + z\zeta_y)} \cdot$$

$$e^{\Omega(z) - \Omega_0(z)} \tag{24}$$

with $\Omega(z)$ defined by

$$\Omega(z) = \int_0^z \tilde{w}(\xi_x, \xi_y, \zeta_x, \zeta_y, z')\, dz' \tag{25}$$

If, instead of an unidirectional beam, we have a pencil of light with a Gaussian angular distribution $I_0\, e^{-\beta(\phi_{x_0}^2 + \phi_{y_0}^2)}$ produced by the laser source, the intensity $I(x, y, z, \phi_x, \phi_y)$ is given by

$$I(x, y, z, \phi_x, \phi_y) = \iint_{-\infty}^{\infty} d\phi_{x_0}\, d\phi_{y_0}\, F(\phi_x, \phi_y, x, y, z)\, I_0\, e^{-\beta(\phi_{x_0}^2 + \phi_{y_0}^2)}$$

$$= \frac{I_0}{(2\pi)^4} \int_{-\infty}^{\infty} (4)\; d\xi_x d\xi_y d\zeta_x d\zeta_y\; e^{-i(\xi_x\phi_x + \xi_y\phi_y + \zeta_x x + \zeta_y y)} \cdot$$

$$e^{\Omega(z) - \Omega_0(z)} \int_{-\infty}^{\infty} (2)\; d\phi_{x_0}\, d\phi_{y_0}\; e^{-\beta(\phi_{x_0}^2 + \phi_{y_0}^2)} \cdot e^{i\phi_{x_0}(\xi_x + z\zeta_x) + i\phi_{y_0}(\xi_y + z\zeta_y)}$$

$$\tag{26}$$

Since

$$\int_{-\infty}^{\infty} (2)\, d\phi_{x_0} d\phi_{y_0}\, e^{i\phi_{x_0}(\xi_x + z\zeta_x) + i\phi_{y_0}(\xi_y + z\zeta_y)}\, e^{-\beta(\phi_{x_0}^2 + \phi_{y_0}^2)}$$

$$= \frac{\pi}{\beta} \exp\left[- \frac{(\xi_x + z\zeta_x)^2 + (\xi_y + z\zeta_y)^2}{4\beta} \right] \tag{27}$$

we get for $I(\vec{r}, \vec{s})$ the expression

$$I(x, y, z, \phi_x, \phi_y) = \frac{I_0}{(2\pi)^4} \int_{-\infty}^{\infty} (4)\, d\xi_x d\xi_y d\zeta_x d\zeta_y$$

$$e^{-i(\xi_x\phi_x + \xi_y\phi_y + \zeta_x x + \zeta_y y)}\, e^{\Omega(z) - \Omega_0(z)}$$

$$\frac{\pi}{\beta} \exp\left[- \frac{(\xi_x + z\zeta_x)^2 + (\xi_y + z\zeta_y)^2}{4\beta} \right] \tag{28}$$

Eq. (28) is the required result based on which the intensity of a collimated laser beam after undergoing multiple small angle scattering can be obtained.

To proceed further, we make two simplifying assumptions as in (Ref. 2). Let us consider the medium to be homogeneous. It then follows that

$$\Omega_0(z) = \int_0^z \sigma(z')\, dz' = \sigma z \tag{29}$$

and

$$\Omega(z) = \int_0^z \tilde{w}(\xi_x, \xi_y, \zeta_x, \zeta_y, z')\, dz'$$

$$\int_0^z dz' \int_{-\infty}^{\infty} (4)\ d^{i(\xi_x \bar{\phi}_x + \xi_y \bar{\phi}_y + \zeta_x \bar{x} + \zeta_y \bar{y})}$$

$$p(\bar{\phi}_x, \bar{\phi}_y)\ \delta[\bar{x} - (z - z')\bar{\phi}_x]\ \delta[\bar{y} - (z - z')\bar{\phi}_y] d\bar{\phi}_x d\bar{\phi}_y d\bar{x} d\bar{y}$$

$$= \int_0^z dz' \int_{-\infty}^{\infty} (2)\ e^{i\bar{\phi}_x[\xi_x + (z - z')\zeta_x] + i\bar{\phi}_y[\xi_y + (z - z')\zeta_y]}$$

$$\frac{\sigma_s}{4\pi}\ p(\bar{\phi}_x, \bar{\phi}_y) d\bar{\phi}_x d\bar{\phi}_y \tag{30}$$

In addition, we will assume that the phase scattering function can be approximated by a Gaussian

$$p(\bar{\phi}_x, \bar{\phi}_y) = W\ e^{-\alpha(\bar{\phi}_x^2 + \bar{\phi}_y^2)} \tag{31}$$

This is true for the scattering of visible and near-infrared radiation by haze or fog. Substituting (31) into (30) we have

$$\Omega(z) = \frac{\sigma_s}{4} \int_0^z W\ dz' \int_{-\infty}^{\infty} (2)\ d\bar{\phi}_x d\bar{\phi}_y\ e^{i\bar{\phi}_x [\xi_x + (z - z')\zeta_x]}$$

$$e^{i\bar{\phi}_y [\xi_y + (z-z')\zeta_y]}\quad e^{-\alpha(\bar{\phi}_x^2 + \bar{\phi}_y^2)} \tag{32}$$

We can again apply the result of (27) to evaluate the integrals of $\bar{\phi}_x$, $\bar{\phi}_y$ in $\Omega(z)$:

$$\Omega(z) = \frac{\sigma_s}{4\pi}\ \frac{\pi}{\alpha}\ W \int_0^z \exp\left\{ - \frac{(\xi_x + (z - z')\zeta_x)^2 + (\xi_y + (z - z')\zeta_y)^2}{4\alpha} \right\} dz' \tag{33}$$

Thus, we can rewrite (28) as follows:

$$I(x, y, z, \phi_x, \phi_y) = -\frac{I_0}{(2\pi)^4} \frac{\pi}{\beta} e^{-\sigma z} \int_{-\infty}^{\infty} (4)\, d\vec{\xi} d\vec{\zeta}\ e^{-i(\vec{\xi}\cdot\vec{\phi} + \vec{\zeta}\cdot\vec{r}_\perp)}$$

$$\exp\left\{ \frac{\sigma_s}{4\pi} \frac{\pi W}{\alpha} \int_0^z \exp\left[- \frac{|\vec{\xi} + (z - z')\vec{\zeta}|^2}{4\alpha} \right] dz' \right\} \exp\left[- \frac{(\vec{\zeta} + z\vec{\zeta})^2}{4\beta} \right] \tag{34}$$

where $\vec{\xi} = (\xi_x, \xi_y)$, $\vec{\zeta} = (\zeta_x + \zeta_y)$, $\vec{\phi} = (\phi_x, \phi_y)$ and $\vec{r}_\perp = (x, y)$. In the above expression, we can carry out analytically either the inverse Fourier transform or the integration over z. For the evaluation of the lowest order scattering terms, it is advantageous to carry out the inverse Fourier transform analytically. On the other hand, to evaluate the contribution from all the high-order scattering terms to $I(\vec{r}, \vec{s})$, the integration over z gives a more convenient result.

To perform the inverse Fourier transform we expand the factor

$$\exp\left\{ \frac{\sigma_s}{4\pi} \frac{\pi W}{\alpha} \int_0^z \exp\left[- \frac{|\vec{\xi} + (z - z')\vec{\zeta}|^2}{4\alpha} \right] \right\}$$

$$= \sum_{m=0}^{\infty} \sigma_s^m \frac{1}{m!} \int_0^z (m) dz_1 . . dz_m \exp\left\{ - \frac{|\vec{\xi} + (z - z')\vec{\zeta}|^2}{4\alpha} - \frac{|\vec{\xi} + (z - z_2)\vec{\zeta}|^2}{4\alpha} - \cdots - \frac{|\vec{\xi} + (z - z_m)\vec{\zeta}|^2}{4\alpha} \right\}$$

$$= \sum_{m=0}^{\infty} \sigma_s^m \frac{1}{m!} \int_0^z (m) dz_1 . . dz_m \exp\left\{ - \frac{1}{4\alpha} \left[m|\vec{\xi}|^2 + 2\vec{\xi}\cdot\vec{\zeta}\, Z_1(m) + |\zeta|^2 Z_2(m) \right] \right\} \tag{35}$$

where $Z_1(m) = \sum_{j=1}^{m} (z - z_j)$, (36)

and $Z_2(m) = \sum_{j=1}^{m} (z - z_j)^2$ (37)

Substituting (35) into (34), we obtain the following series expansion

$$I(x, y, z, \phi_x, \phi_y) = \sum_{m=0}^{\infty} I_m(x, y, z, \phi_x, \phi_y) \tag{38}$$

and $I_m(x, y, z, \phi_x, \phi_y)$ is defined as

$$I_m(x, y, z, \phi_x, \phi_y) = \frac{I_0}{(2\pi)^4} \frac{\pi}{\beta} e^{-\sigma z} \frac{\sigma_s^m}{m!} \int_{-\infty}^{\infty} (4)\, d\vec{\xi} d\vec{\zeta}$$

$$e^{-i(\vec{\xi}\vec{\phi} + \vec{\zeta}\,\vec{r}_\perp)} \cdot \int_0^z (m) dz_1 .\ . dz_m \left(\exp\left\{ - \frac{1}{4\alpha} \left[m|\vec{\xi}|^2 + 2\vec{\xi}\vec{\zeta}\, Z_1(m) + \right.\right.\right.$$

$$\left.\left.\left. |\vec{\zeta}|\, Z_2(m) \right] \right\} . \exp\left\{ - \frac{1}{4\beta} \left[|\vec{\xi}|^2 + 2z(\vec{\xi}\vec{\zeta}) + z^2 |\vec{\xi}|^2 \right] \right\} \right) \tag{39}$$

We interchange the order of integration so that

$$I_m(x, y, z, \phi_x, \phi_y) = \frac{I_0}{(2\pi)^4} \frac{\pi}{\beta} e^{-\sigma z} \frac{\sigma_s^m}{m!} \int_0^z (m) dz_1 .\ . dz_m \ .$$

$$\int_{-\infty}^{\infty} (4)\, d\vec{\xi} d\vec{\zeta}\ \exp\left\{ - \left[|\vec{\xi}|^2 \left(\frac{m}{4\alpha} + \frac{1}{4\beta} \right) + 2\vec{\xi}\vec{\zeta} \left(\frac{Z_1(m)}{4\alpha} + \frac{z}{4\beta} \right) + \right.\right.$$

$$\left.\left. |\vec{\xi}|^2 \left(\frac{Z_2(m)}{4\beta} + \frac{z^2}{4\beta} \right) \right] \right\} e^{-i(\vec{\xi}\vec{\phi} + \vec{\zeta}\vec{r}_\perp)} \tag{40}$$

It can be shown that

$$\int_{-\infty}^{\infty} (4)\ d\vec{\xi}d\vec{\zeta}\ e^{-i(\vec{\xi}\vec{\phi} + \vec{\zeta}\vec{r}_\perp)} \quad \exp\left\{ - \left[|\vec{\xi}|^2 \left(\frac{m}{4\alpha} + \frac{1}{4\beta} \right) + \right.\right.$$

$$\left.\left. 2(\vec{\xi}\vec{\zeta}) \left(\frac{Z_1(m)}{4\alpha} + \frac{z}{4\beta} \right) + |\vec{\xi}|^2 \left(\frac{Z_2(m)}{4\alpha} + \frac{z^2}{4\beta} \right) \right] \right\}$$

$$= \frac{(2\pi)^2}{\Delta(m)} \quad \exp\left\{ \frac{1}{\Delta(m)} \quad -\left(\frac{Z_2(m)}{4\alpha} + \frac{z^2}{4\beta} \right) |\vec{\phi}|^2 + 2\left(\frac{Z_1(m)}{4\alpha} + \frac{z}{4\beta} \right) (\vec{\phi} \cdot \vec{r}_\perp) \right.$$

$$\left. -\left(\frac{m}{4\alpha} + \frac{1}{4\beta} \right) |\vec{r}_\perp|^2 \right] \Bigg\} \tag{41}$$

with $\Delta(m)$ given by

$$\Delta(m) = 4 \left(\frac{m}{4\alpha} + \frac{1}{4\beta} \right) \left(\frac{Z_2(m)}{4\alpha} + \frac{z^2}{4\beta} \right) - \left(\frac{Z_1(m)}{4\alpha} + \frac{z}{4\beta} \right)^2 \tag{42}$$

Hence, the contribution to $I(\vec{r}, \vec{s})$ by the m^{th} order scattering is

$$I_m(x, y, z, \phi_x, \phi_y) = \frac{I_0}{(2\pi)^4} \quad \frac{\pi}{\beta} \quad e^{-\sigma z} \quad \frac{\sigma_s^m}{m!} \quad (2\pi)^2 \quad .$$

$$\int_0^z (m)\ dz_1 .\ .\ dz_m \quad \frac{1}{\Delta(m)} \quad \exp\left\{ \frac{1}{\Delta(m)} \left[- \left(\frac{Z_2(m)}{4\alpha} + \frac{z^2}{4\beta} \right) |\vec{\phi}|^2 \right.\right.$$

$$\left.\left. + \ 2 \left(\frac{Z_1(m)}{4\alpha} + \frac{z}{4\beta} \right) (\vec{\phi} . \vec{r}_\perp) - \left(\frac{m}{4\alpha} + \frac{1}{4\beta} \right) |\vec{r}_\perp|^2 \right] \right\} \tag{43}$$

the numerical evaluation of which involves the computation of a multiple integration of m^{th} order.

Alternatively we may integrate analytically the following integral occurring in (34)

$$\int_0^z \exp\left[-\frac{|\vec{\xi} + (z - z')\zeta|^2}{4\alpha}\right] dz'$$

$$= \int_0^z \exp\left\{-\frac{1}{4\alpha}\left[|\vec{\xi}|^2 + 2(z - z')\ (\vec{\xi}\vec{\zeta}) + (z - z')^2|\vec{\zeta}|^2\right]\right\} dz'$$

$$= \int_0^z \exp\left\{-\frac{1}{4\alpha}\left[z'^2\vec{\zeta}^2 - 2z'(\vec{\xi}\vec{\zeta} + z\vec{\zeta}^2) + (\vec{\xi}^2 + 2z\vec{\xi}\vec{\zeta} + z^2\vec{\zeta}^2)\right]\right\} dz'$$

$$= \sqrt{\frac{\pi\alpha}{\vec{\xi}^2}}\ \exp\left\{-\frac{1}{4\alpha}\ \frac{\vec{\xi}^2\vec{\zeta}^2 - (\vec{\xi}\vec{\zeta})^2}{\vec{\zeta}^2}\right\}\cdot\left[\operatorname{erf}\left(\frac{(\vec{\xi}\vec{\zeta}) + z\vec{\zeta}^2}{\sqrt{4\alpha}\ |\vec{\zeta}|}\right) - \operatorname{erf}\left(\frac{\vec{\xi}\vec{\xi}}{\sqrt{4\alpha}\ |\vec{\zeta}|}\right)\right] \tag{44}$$

from which, the intensity $I(\vec{r}, \vec{s})$ can be expressed as an inverse Fourier transform given by:

$$I(x, y, z, \phi_x, \phi_y) = \frac{I_0}{(2\pi)^4} \int_{-\infty}^{\infty} (4)\ d\vec{\xi} d\vec{\zeta}\ e^{-i(\vec{\xi}\vec{\phi} + \vec{\zeta}\vec{r}_1)}\ \frac{\pi}{\beta}\ e^{-\sigma z}$$

$$\exp\left[-\frac{(\vec{\xi} + z\vec{\zeta})^2}{4\beta}\right] \exp\left\{\frac{\sigma_s\sqrt{\pi\alpha}}{|\vec{\zeta}|}\ \exp\left\{-\frac{1}{4\alpha}\ \frac{\vec{\xi}^2\vec{\zeta}^2 - (\vec{\xi}\vec{\zeta})^2}{\vec{\zeta}^2}\right\}\cdot\right.$$

$$\left.\left[\operatorname{erf}\left(\frac{\vec{\xi}\vec{\zeta} + z\vec{\zeta}^2}{\sqrt{4\alpha}\ |\vec{\zeta}|}\right) - \operatorname{erf}\left(\frac{\vec{\xi}\vec{\zeta}}{\sqrt{4\alpha}\ |\vec{\zeta}|}\right)\right]\right\} \tag{45}$$

Although equation (43) when substituted into (38) gives the same result as (45), however, for numerical computation (43) can be used more conveniently for low-order scattering. We recall that in the series expansion the number of integrations to be carried out numerically is equal to the order of scattering being considered. To find the

contribution from all the high-order multiple scattering terms, the result in (45) can be used and the four-dimensional integration have to be evaluated numerically. Despite the fact that (45) can be expanded into an infinite series with each term corresponding to a specific order of scattering, the application of such an expression to calculate the lower order terms is not as convenient as using (43). Even for the first-order term, a four-dimensional integration over an infinite domain has to be computed. It can be seen from the explicit expression, or from physical considerations, that, for the scattering of a tight beam by an anisotropic medium, the beam will not be significantly broadened by lower order multiple scattering. Therefore, the integrands corresponding to these terms obtained from expanding (45) tend to spread out in the $\vec{\xi}$, $\vec{\zeta}$ space making numerical computation laborious.

2.3 Solution of the Radiative Transfer Equation in the Small-Angle Approximation

In the absence of sources inside the medium, the radiative transfer equation is given in its general form by

$$\vec{s}.\nabla_{\vec{r}}\, I(\vec{r}, \vec{s}) + \sigma I(\vec{r}, \vec{s}) = \frac{\sigma_s}{4\pi} \iint p(\vec{s}, \vec{s})\, I(\vec{r}, \vec{s}) d\Omega_{\vec{s}}' \tag{46}$$

For simplicity, we have taken the medium to be homogeneous such that the phase function is independent of the position vector $\vec{r}$. If $\vec{i}$, $\vec{j}$, $\vec{k}$ are unit vectors along x, y and z axes respectively, the vector $\vec{s}$ can be written in terms of the polar angles (θ, ϕ) as

$$\vec{s} = \sin\theta \cos\phi\, \vec{i} + \sin\theta \sin\phi\, \vec{j} + \cos\phi\, \vec{k} \tag{47}$$

and $$\vec{s}' = \sin\theta' \cos\phi'\, \vec{i} + \sin\theta' \sin\phi'\, \vec{j} + \cos\theta'\, \vec{k} \tag{48}$$

Let Θ be the angle between $\vec{s}$ and $\vec{s}'$. From the identity

$$\cos \Theta = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos (\phi - \phi') \tag{49}$$

and the small-angle approximation

$$\sin\theta \cos\phi = \theta\cos\phi = \phi_x$$

$$\sin\theta \sin\phi = \theta\sin\phi = \phi_y,$$

we obtain

$$\begin{aligned} \cos \Theta &= \left(1 - \frac{\theta^2}{2}\right)\left(1 - \frac{\theta^2}{2}\right) + \theta\theta' \ (\cos\phi \cos\phi' + \sin\phi \sin\phi') \\ &= 1 - \frac{\theta^2}{2} - \frac{\theta'^2}{2} + \theta\theta' \ (\cos\phi \cos\phi' + \sin\phi \sin\phi') \\ &= 1 - \frac{1}{2} \ (\phi_x - \phi_x')^2 - \frac{1}{2} (\phi_y - \phi_y')^2 \end{aligned} \tag{50}$$

Thus, the phase function $p(\vec{s}, \vec{s}) = p(\Theta)$ in the small-angle approximation is a function of the differences of the projected angles, i.e. $\phi_x - \phi_x'$ and $\phi_y - \phi_y'$ only. This enables us to write the radiative transfer equation in the form

$$\phi_x \frac{\partial I}{\partial x} + \phi_y \frac{\partial I}{\partial y} + \frac{\partial I}{\partial z} + \sigma I$$

$$= \frac{\sigma_s}{4\pi} \int\int_{-\infty}^{\infty} d\phi_x' d\phi_y' \ p(\phi_x - \phi_x', \phi_y - \phi_y') \ I(x, y, z, \phi_x', \phi_y') \tag{51}$$

Let $\tilde{I}(\vec{\xi}, \vec{\zeta}, z)$ and $\tilde{p}(\vec{\xi})$ be the Fourier transforms of $I(\vec{r}, \vec{s})$ and $p(\Theta)$ respectively such that

$$\tilde{I}(\vec{\xi}, \vec{\zeta}, z) = \int_{-\infty}^{\infty} (4) \ d\phi_x d\phi_y dx dy \ I(\vec{r}, \vec{s}) \ e^{i(\vec{\xi}\vec{\phi} + \vec{\zeta}\vec{r})} \tag{52}$$

$$\tilde{p}(\vec{\xi}) = \int_{-\infty}^{\infty} (2)\, d\phi_x d\phi_y p(\phi_x, \phi_y)\, e^{i\vec{\xi}\vec{\phi}} \tag{53}$$

and $\vec{r}_\perp = (x,y)$.

Substituting (52) and (53) into (51) we obtain the following linear partial differential equation for $\tilde{I}(\vec{\xi}, \vec{\zeta}, z)$:

$$- \zeta_x \frac{\partial}{\partial \xi_x} + \zeta_y \frac{\partial}{\partial \zeta_y} \quad \tilde{I}(\vec{\xi}, \vec{\zeta}, z) + \frac{\partial}{\partial z} \tilde{I}(\vec{\xi}, \vec{\zeta}, z) + \sigma \tilde{I}(\vec{\xi}, \vec{\zeta}, z)$$

$$= \frac{\sigma_s}{4\pi} \tilde{p}(\vec{\xi})\, \tilde{I}(\vec{\xi}, \vec{\zeta}, z) \tag{54}$$

The solution of the above equation can be obtained using the standard theory which we will describe only briefly. Let $\tilde{I}(\vec{\xi}, \vec{\zeta}, z)$ be given in the form of an implicit solution

$$\Psi(\xi_x, \xi_y, z, \tilde{I}) = \text{Constant} \tag{55}$$

Since

$$\frac{\partial \tilde{I}}{\partial \xi_x} = - \frac{(\partial\Psi/\partial\xi_x)}{(\partial\Psi/\partial\tilde{I})}, \quad \frac{\partial \tilde{I}}{\partial \xi_y} = - \frac{(\partial\Psi/\partial\xi_y)}{(\partial\Psi/\partial\tilde{I})}$$

and

$$\frac{\partial \tilde{I}}{\partial z} = \frac{-(\partial\Psi/\partial z)}{(\partial\Psi/\partial\tilde{I})}$$

these relations, together with (54) give the following equation in Ψ:

$$-\zeta_x \frac{\partial\Psi}{\partial\xi_x} - \zeta_y \frac{\partial\Psi}{\partial\xi_y} + \frac{\partial\Psi}{\partial z} = \left(\frac{\sigma_s}{4\pi} \tilde{p}(\vec{\xi}) - \sigma \right) \tilde{I} \frac{\partial\Psi}{\partial\tilde{I}} \tag{56}$$

and is equivalent to the set of differential equations

$$-\frac{d\xi_x}{\zeta_x} = -\frac{d\xi_y}{\zeta_y} = dz = \frac{d\tilde{I}}{\frac{\sigma_s \tilde{p}}{4\pi} - \sigma\,\tilde{I}} \qquad (57)$$

On integration (57) gives

$$\xi_x + \zeta_y z = C_x\ ,\ \xi_y + \zeta_y z = C_y \qquad (58)$$

and

$$\tilde{I} = C \exp\int\left[\frac{\sigma_s}{4\pi}\ \tilde{p}(\vec{\xi}) - \sigma\right] dz \qquad (59)$$

where C_x, C_y and C are constants of integration. The general solution of (58) is

$$\Psi(\xi_x, \xi_y, z, \tilde{I}) = \Psi(C_x, C_y, C, z) \qquad (60)$$

The fact that $\Psi(\xi_x, \xi_y, z, \tilde{I})$ = Constant enables us to solve for C such that

$$C = \Phi(C_x, C_y) \qquad (61)$$

and

$$\tilde{I}(\vec{\xi}, \vec{\zeta}, z) = \Phi(\vec{\xi} + z\vec{\zeta}) \exp \int_0^z \left\{ \frac{\sigma_s}{4\pi}\ \tilde{p}\left[\vec{\xi} + (z - z')\vec{\zeta}\right] - \sigma \right\} dz' \qquad (62)$$

Putting $z = 0$ in (62), we obtain

$$\tilde{I}(\vec{\xi}, \vec{\zeta}, 0) = \Phi(\vec{\xi})$$

Therefore, $\Phi(\vec{\xi} + z\vec{\zeta}) = \tilde{I}(\vec{\xi} + z\vec{\zeta}, \vec{\zeta}, 0)$ (63)

Using (63) and the fact that the medium is homogeneous, we can write

$$\tilde{I}(\vec{\xi}, \vec{\zeta}, z) = \tilde{I}(\vec{\xi} + z\vec{\zeta}, \vec{\zeta}, 0)\, e^{-\sigma z}\, e^{\Omega} \tag{64}$$

where

$$\Omega = \int_0^z \frac{\sigma_s}{4\pi}\, \tilde{p}\left[\vec{\xi} + \vec{\zeta}(z - z')\right] dz' \tag{65}$$

It is evident that (64) is identical to (63) obtained from using Wentzel's summation method.

3.0 NUMERICAL RESULTS AND DISCUSSIONS

In this section, we present numerical results of the scattered intensity of a laser beam in an optically dense medium using the analytic expressions developped in the previous section. For these calculations, we assume that the laser beam has an angular divergence equal to approximately 0.1 mrad such that the intensity distribution of the incident beam is given by

$$I(x,y,z = 0, \phi_x, \phi_y) = I_o e^{-\beta(\phi_x^2+\phi_y^2)}\, \delta(x)\, \delta(y) \tag{66}$$

with $\beta = 10^8/\text{rad}^2$. The scattering phase function given by (31) with $\alpha = 10^2/\text{rad}^2$ is used. The normalization condition of the phase function we adopt is

$$\iint p(\phi_x, \phi_y)\, d\phi_x\, d\phi_y = 4\pi$$

and hence, $W = 4\alpha$.

To evaluate the scattered intensity $I_s(x, y, z, \phi_x, \phi_y)$ defined by

$$I_s(x, y, z, \phi_x, \phi_y) = \sum_{m=1}^{\infty} I_m(x, y, z, \phi_x, \phi_y) \tag{67}$$

where $I_m(x, y, z, \phi_x, \phi_y)$ is given by (43), it is necessary to compute numerically the m-fold integrals in I_m. For this purpose, the method of Lyness symmetric integration rule has been used (Ref. 7). A sufficient number of terms of $I_m(x, y, z, \phi_x, \phi_y)$ can be evaluated to give the scattered intensity to any degree of accuracy. Because of the rather rapid increase in the computer time required for evaluating the higher dimensional integrals, our program was stopped when a precision of 5% was obtained. In Fig. 4a to Fig. 4f, the scattered intensity is plotted against the lateral displacement for cases corresponding to $\sigma = \sigma_s = 10^{-5}\text{cm}^{-1}$. Fig. 5a to Fig. 5f give a similar set of plots for $\sigma = \sigma_s = 4 \times 10^{-5}\text{cm}^{-1}$. In these results, the typical number of integrations required for each point was between 3 and 6. The smooth curves connecting the computed points were computer fits with functions possessing two continuous derivatives.

In (Refs. 1 and 2), an approximative procedure was introduced to solve the radiative transfer equation with external sources. This consists of replacing Ω (z) of (33) by the following expression:

$$\Omega^{AS}(z) = \sigma_s \int_0^z 1 - \frac{(\xi x + (z-z')\ \zeta x)^2 + (\xi y + (z-z')\ \zeta y)^2}{4\alpha}\, dz'$$

$$= \sigma_s \left[z - \frac{1}{4\alpha} (\vec{\xi}^2 z + \vec{\xi}\, \vec{\zeta}\, z^2 + \frac{1}{3} \vec{\zeta}^2 z^3) \right] \tag{68}$$

On substituting $\Omega^{AS}(z)$ into (34), the corresponding intensity $I^{AS}(x, y, z, \phi_x, \phi_y)$ can readily be obtained in a close analytic form:

$$I^{AS}(x,\ y,\ z,\ \phi_x,\ \phi_y) = \frac{12\alpha^2\ I_o}{\pi\beta\sigma_s^2\ z^4}\ \exp\left[-\ \frac{\alpha\ \vec{\phi}^2}{\sigma_s\ z}\right]\ \cdot$$

$$\exp\left[-\ \frac{12\alpha}{\sigma_s\ z^3}\ (\vec{\gamma}\ -\ \tfrac{1}{2}\ z\vec{\phi})^2\right] \tag{69}$$

Thus, the scattered intensity becomes

$$I_s^{AS}(x,y,z,\phi_x,\phi_y) = \frac{12\ \alpha^2\ I_o}{\pi\beta\sigma_s^2\ z^4}\ \exp\left[-\ \frac{\alpha\vec{\phi}^2}{\sigma_s\ z} - \frac{12\alpha}{\sigma_s z^3}\ (\vec{\gamma}\ -\ \frac{z}{2}\ \vec{\phi})^2\right]$$

$$-\ I_o e^{-\sigma_s z}\ \delta\ (\vec{\gamma}\ -\ z\vec{\phi}) \tag{70}$$

This approximate expression is supposed to be valid for $\sigma_s\ z >> 1$ and $|\ \vec{\xi} + z\ \vec{\zeta}\ |\ /\ 2\sqrt{\alpha} << 1$. However, from the comparison in Fig. 6, showing the exact result with those obtained using (70) the inadequacy of the above approximation is obvious.

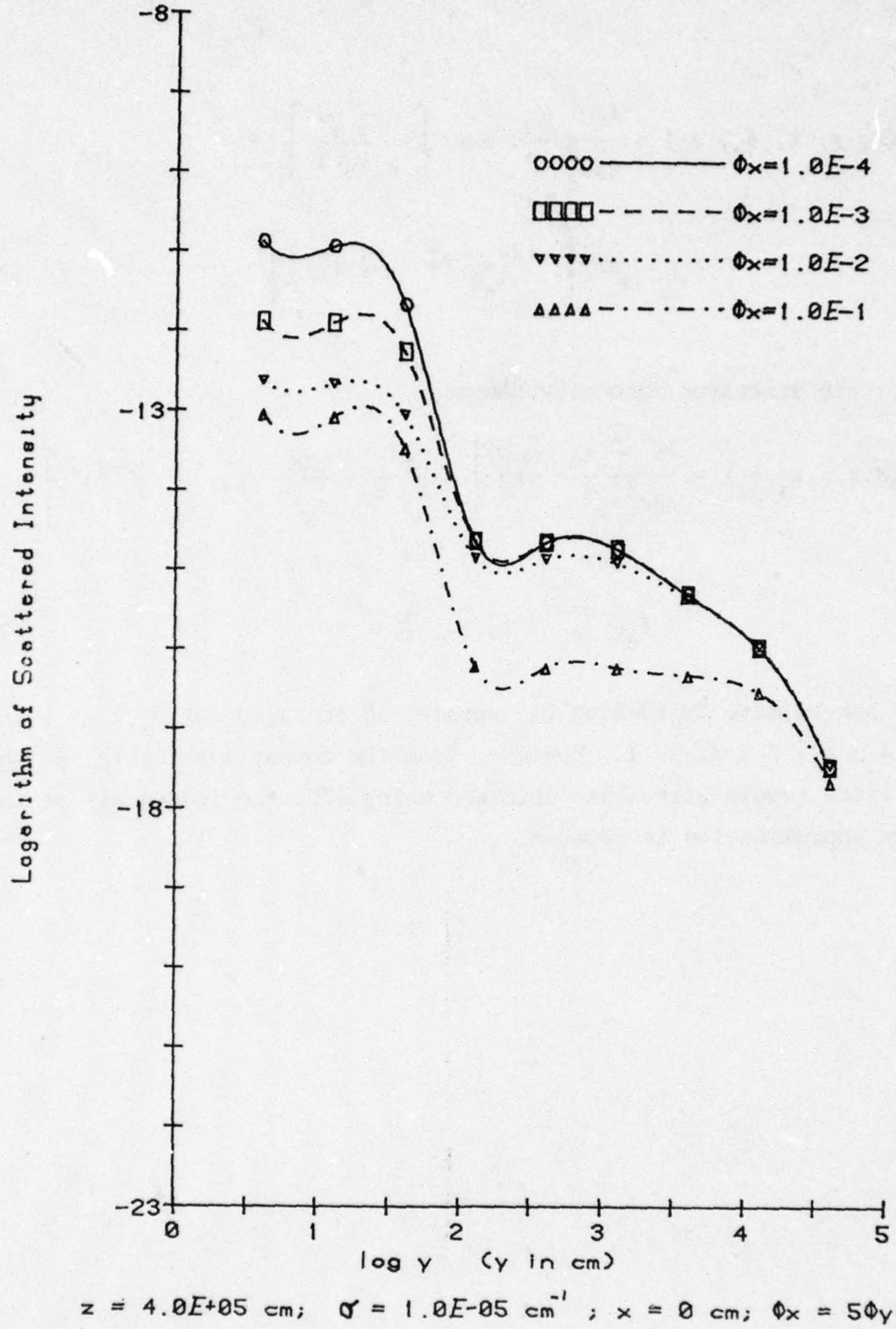

FIGURE 4a - Scattered intensity versus lateral distance x for z = 4×10^{5} cm, $\sigma = 10^{-5}$ cm $^{-1}$, y = 0 and $\phi_x = 5\phi_y$

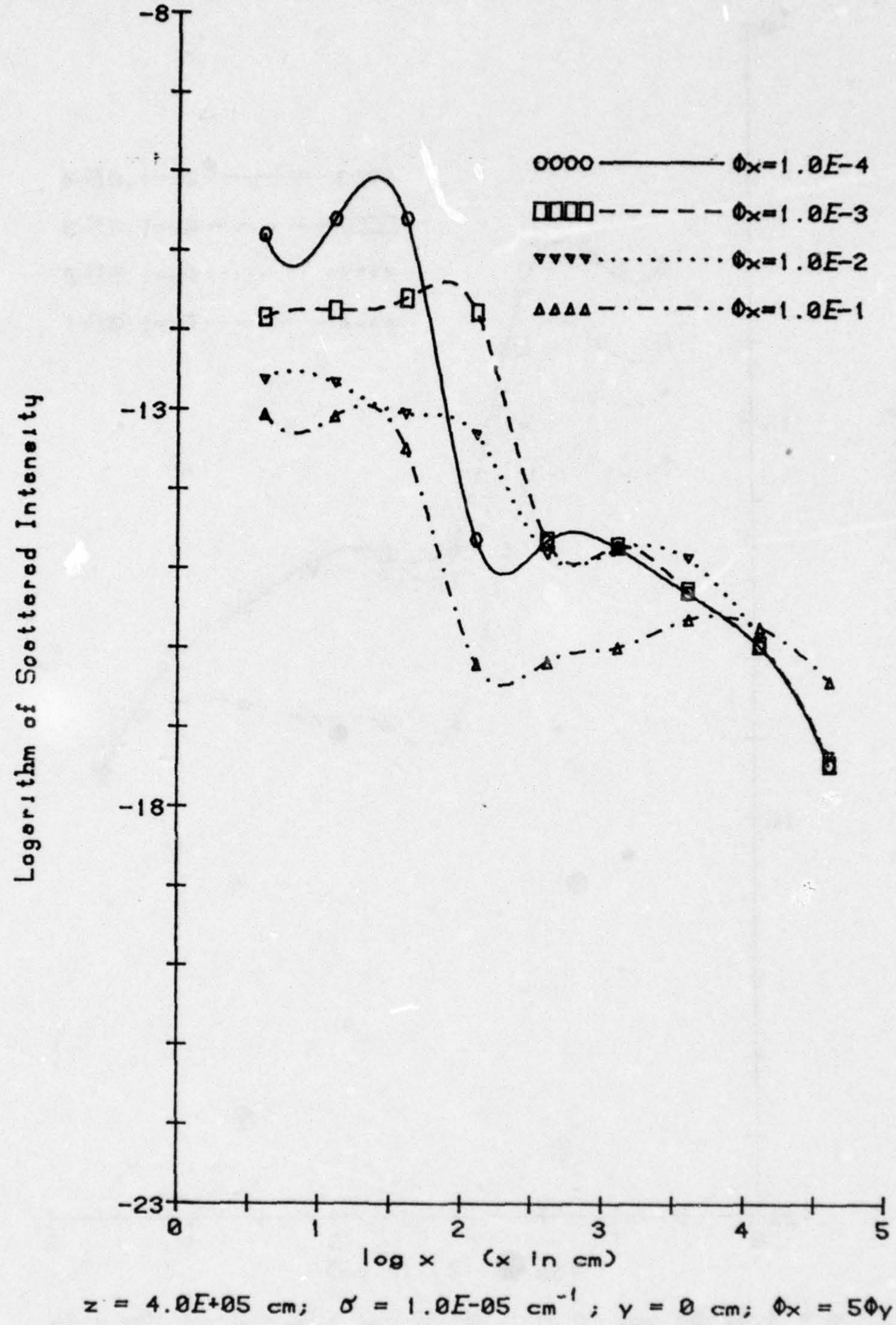

FIGURE 4b - Scattered intensity versus lateral distance y for z = 4×10^5 cm, $\sigma = 10^{-5}$ cm $^{-1}$, x = 0 and $\phi_x = 5\phi_y$

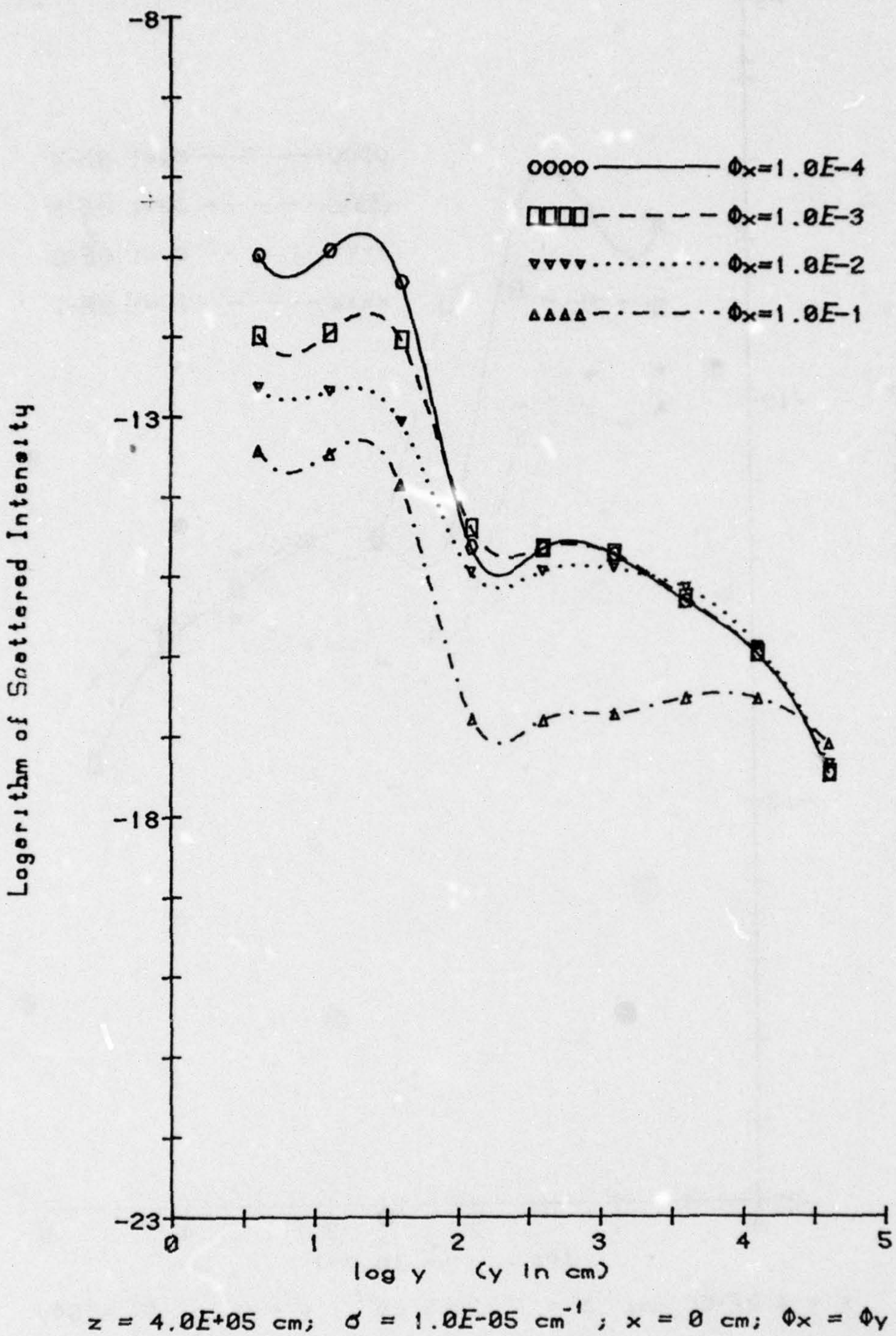

FIGURE 4c - Scattered intensity versus lateral distance y for z = 2×10^5 cm, $\sigma = 10^{-5}$ cm $^{-1}$, x = 0 and $\phi_x = 5\phi_y$

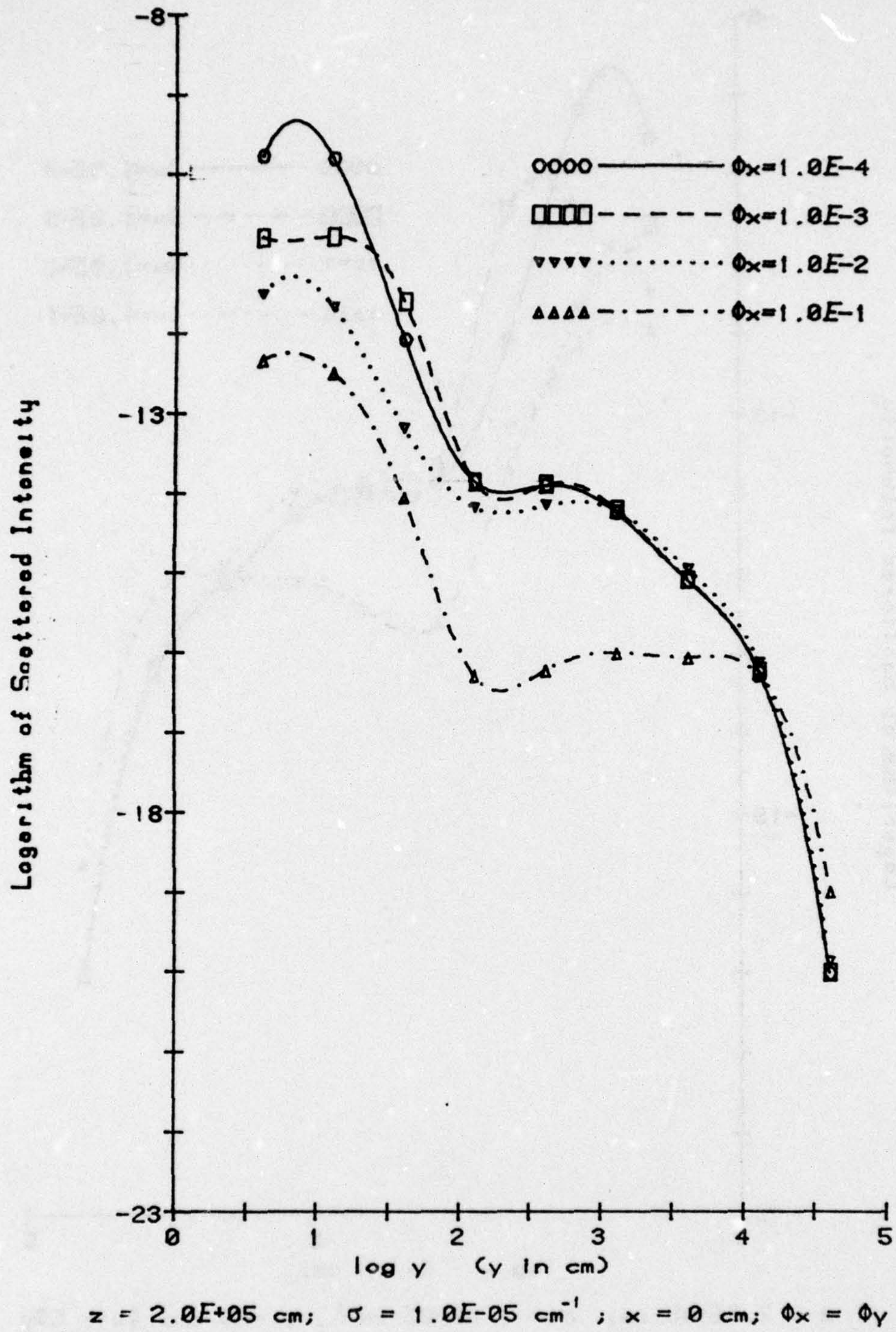

FIGURE 4d - Scattered intensity versus lateral distance y for $z = 4 \times 10^5$ cm, $\sigma = 10^{-5}$ cm $^{-1}$, $x = 0$, $\phi_x = \phi_y$

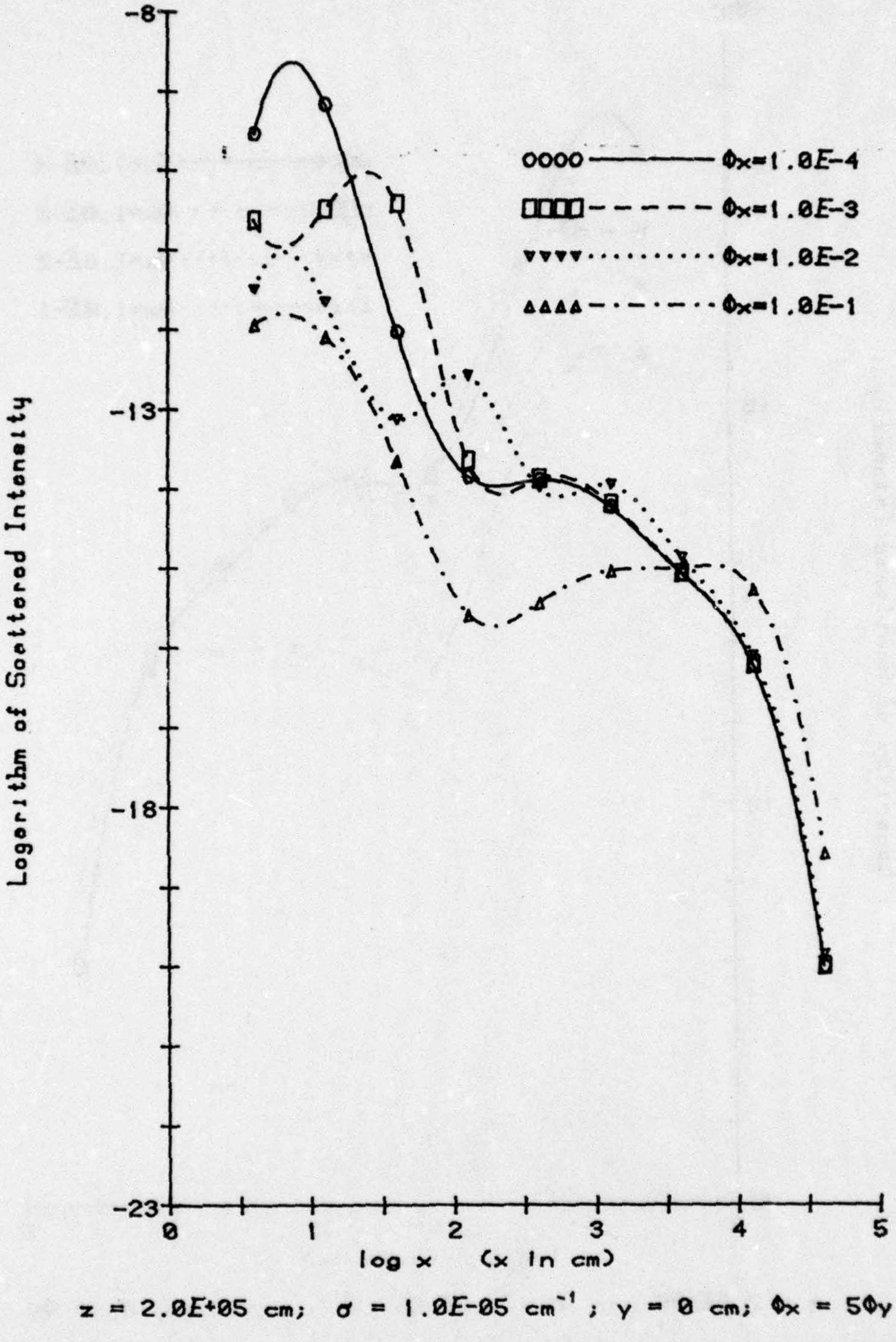

FIGURE 4e - Scattered intensity versus lateral distance x for $z = 2 \times 10^5$ cm, $\sigma = 10^{-5}$ cm $^{-1}$, $y = 0$, $\phi_x = 5\phi_y$

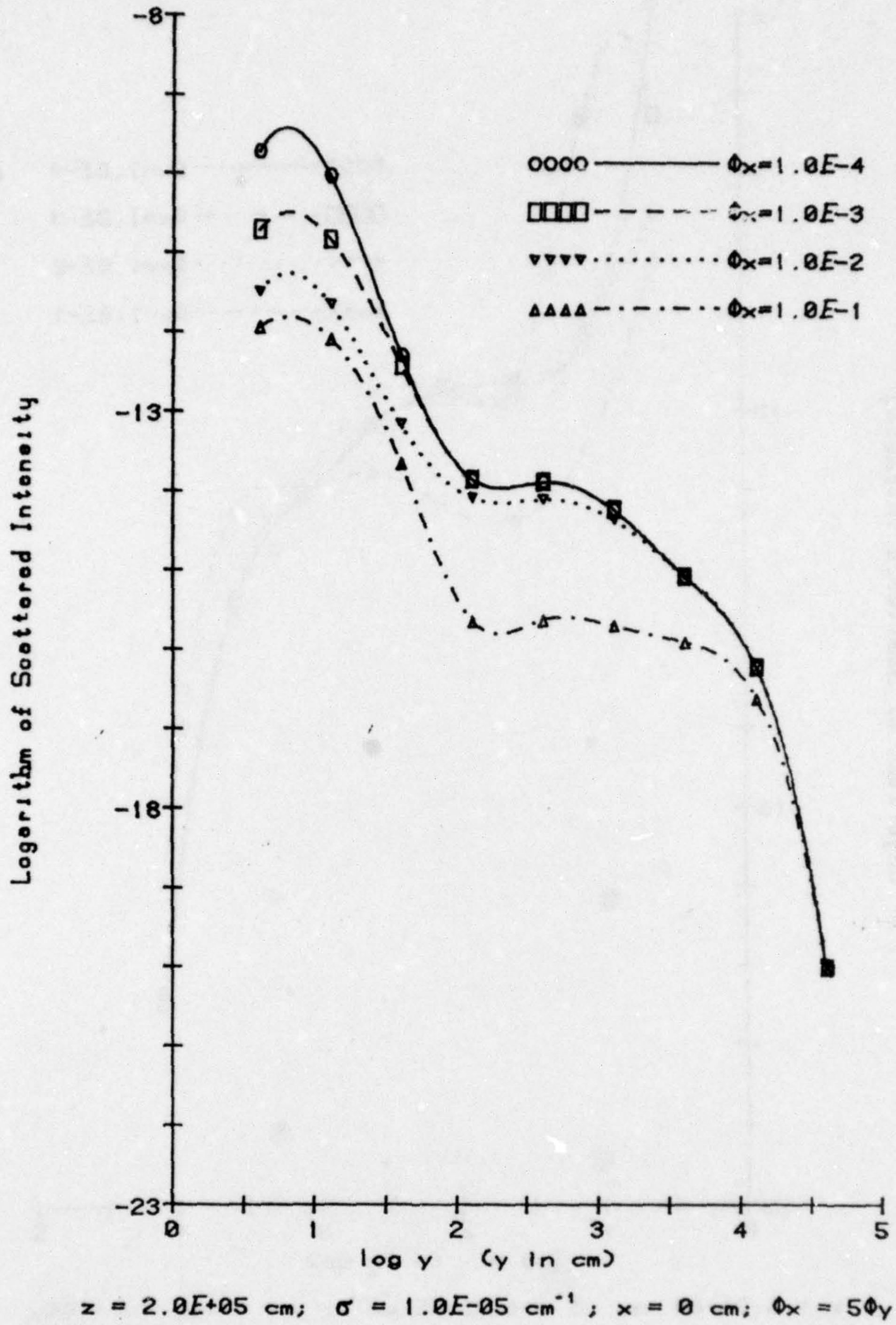

FIGURE 4f - Scattered intensity versus lateral distance y for z = 2×10^5 cm, $\sigma = 10^{-5}$ cm $^{-1}$, x = 0, $\phi_x = \phi_y$

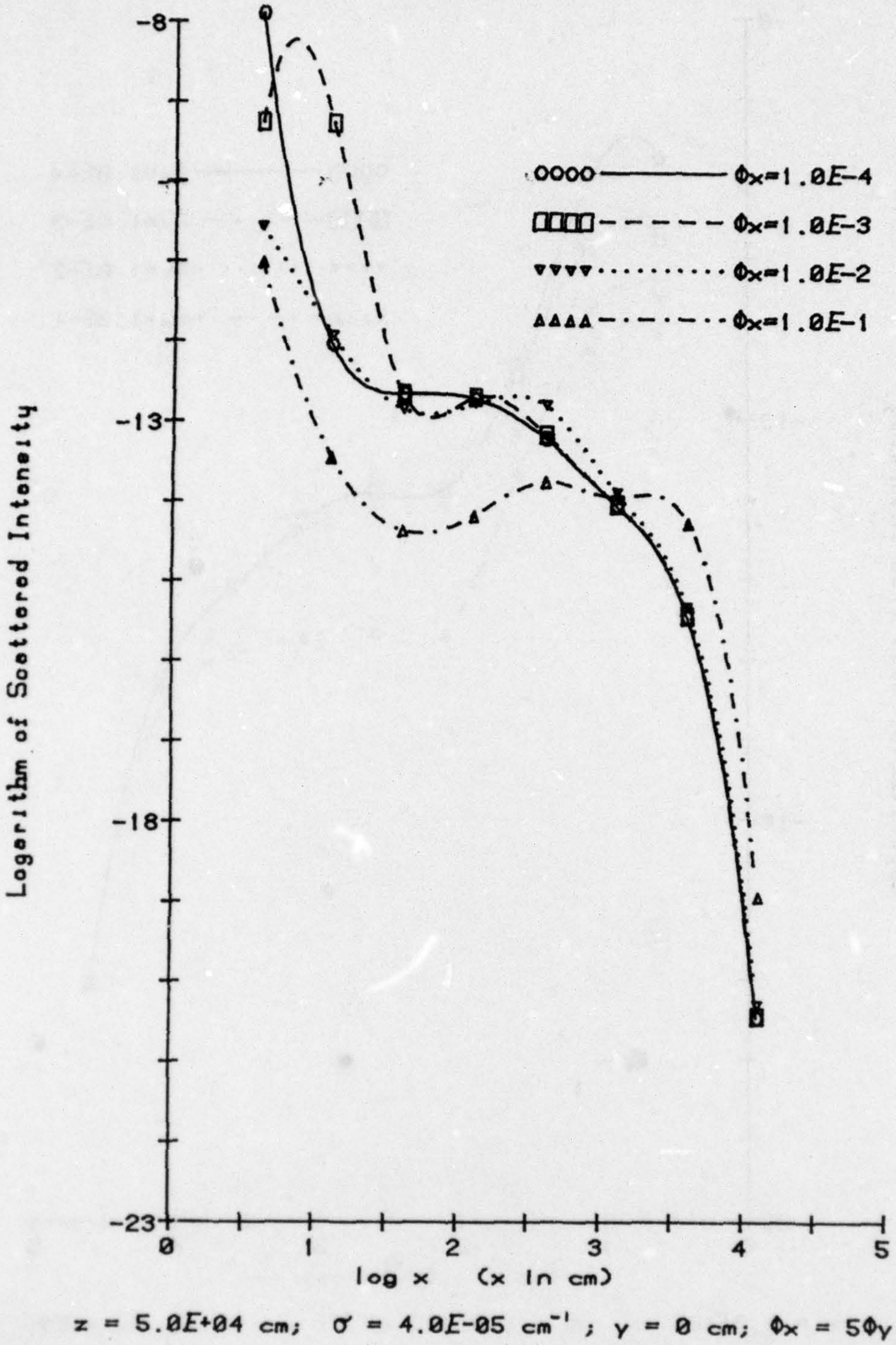

FIGURE 5a - Scattered intensity versus lateral distance x for z = 10^5 cm, $\sigma = 4 \times 10^{-5}$ cm $^{-1}$, y = 0, $\phi_x = 5\phi_y$

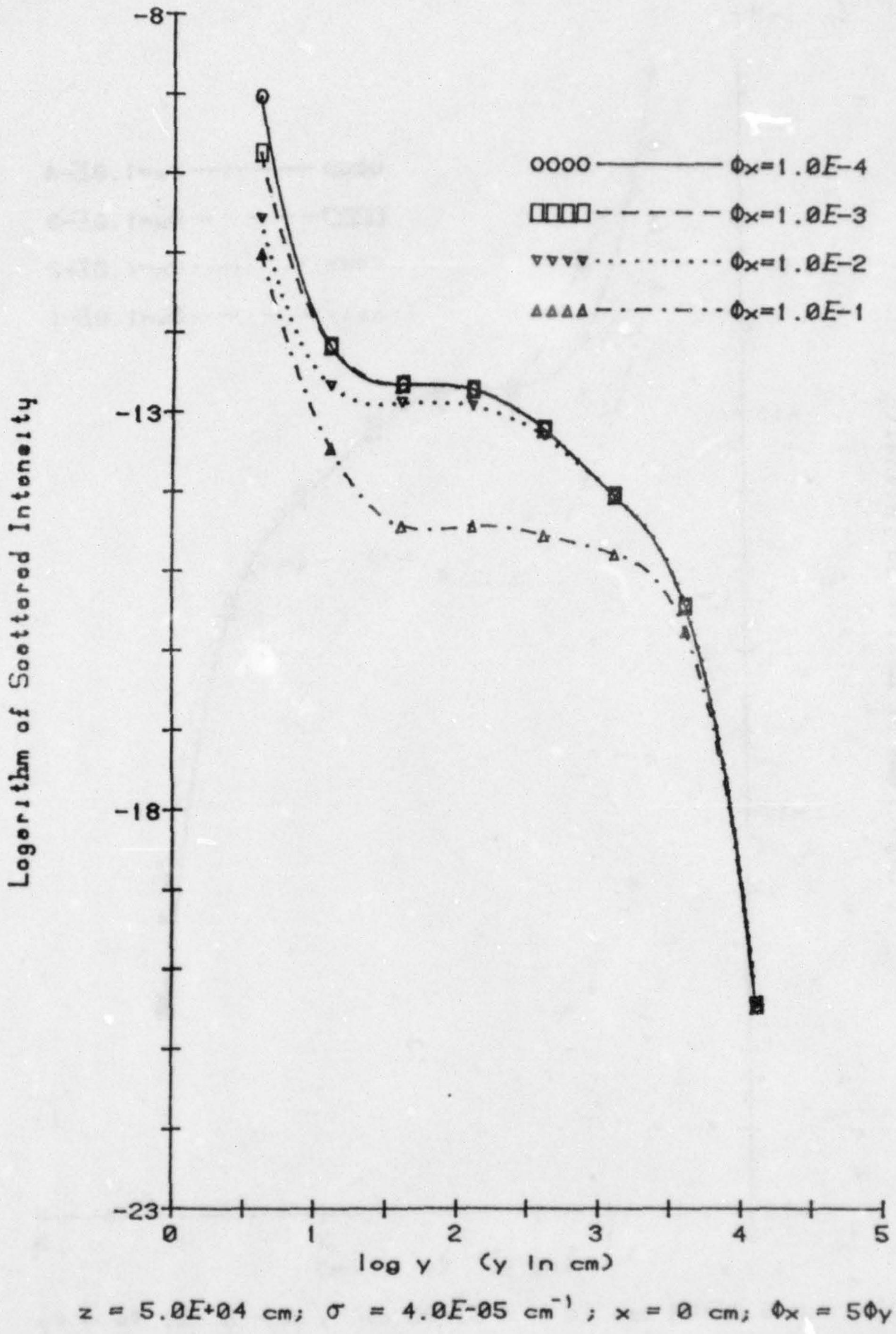

FIGURE 5b - Scattered intensity versus lateral distance y for z = 5×10^4 cm, $\sigma = 4 \times 10^{-5}$ cm $^{-1}$, x = 0, $\phi_x = 5\phi_y$

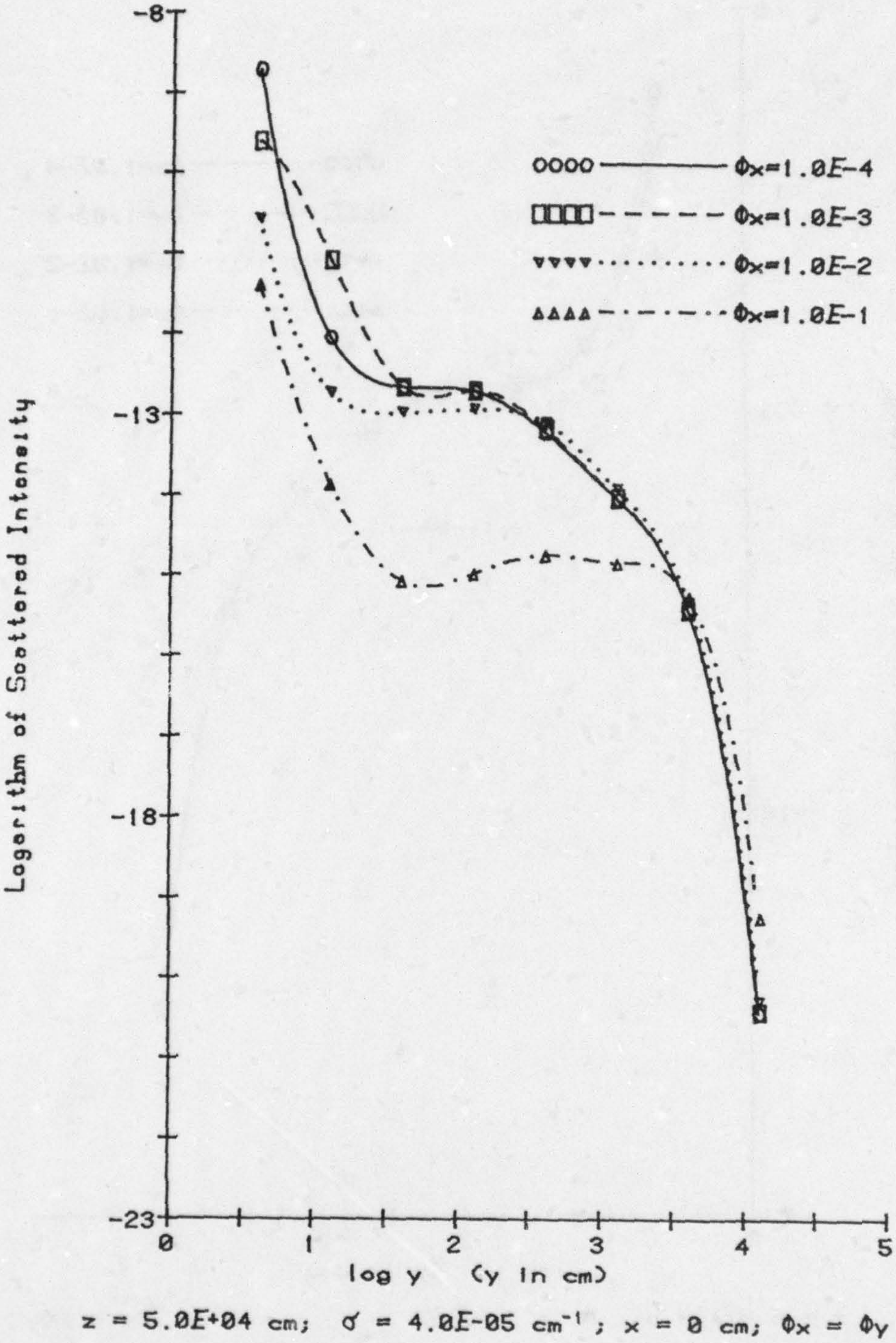

FIGURE 5c - Scattered intensity versus lateral distance y for z = 10^5 cm, $\sigma = 4 \times 10^{-5}$ cm $^{-1}$, x = 0, $\phi_x = \phi_y$

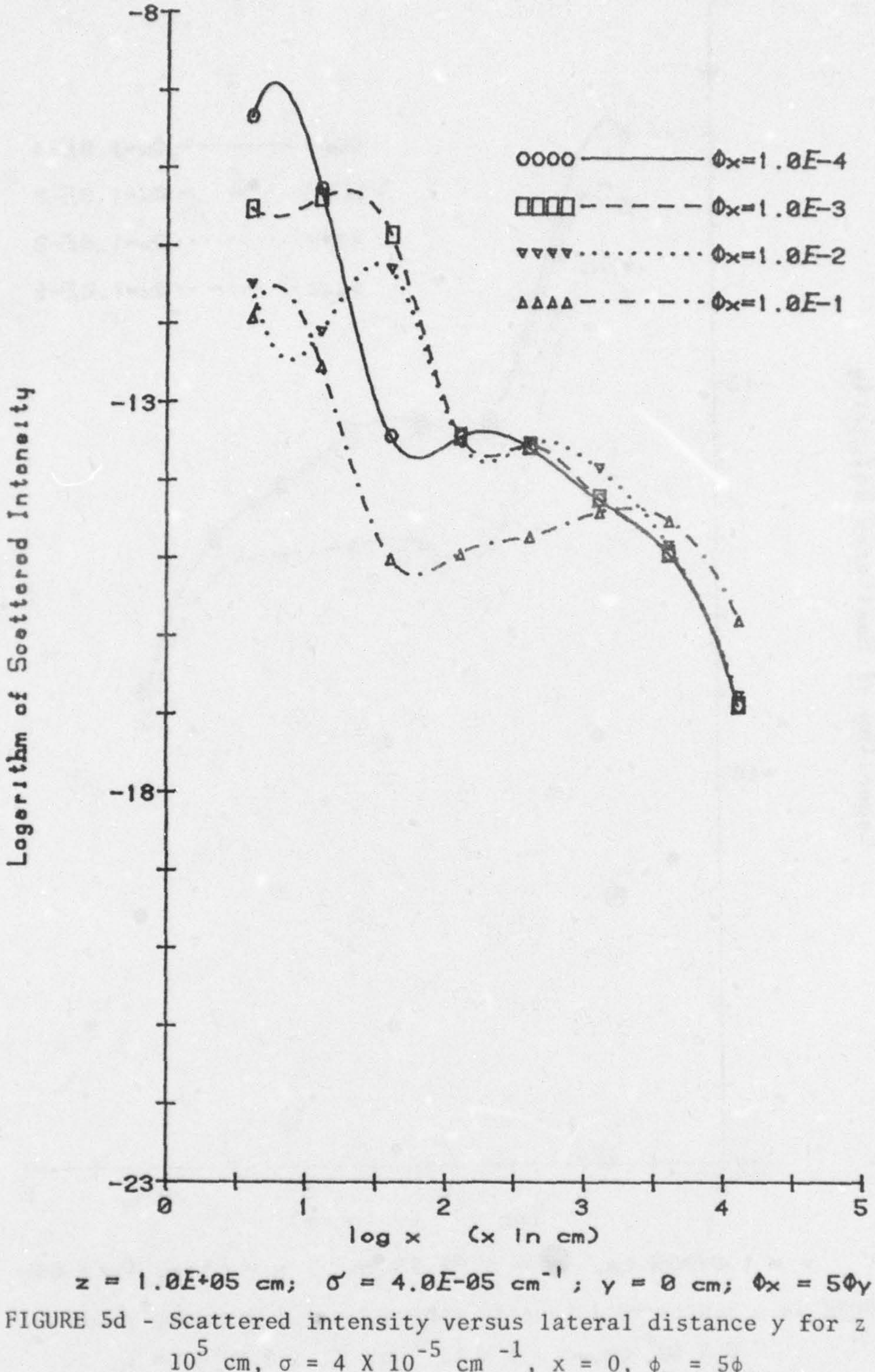

FIGURE 5d - Scattered intensity versus lateral distance y for z = 10^5 cm, $\sigma = 4 \times 10^{-5}$ cm $^{-1}$, x = 0, $\phi_x = 5\phi_y$

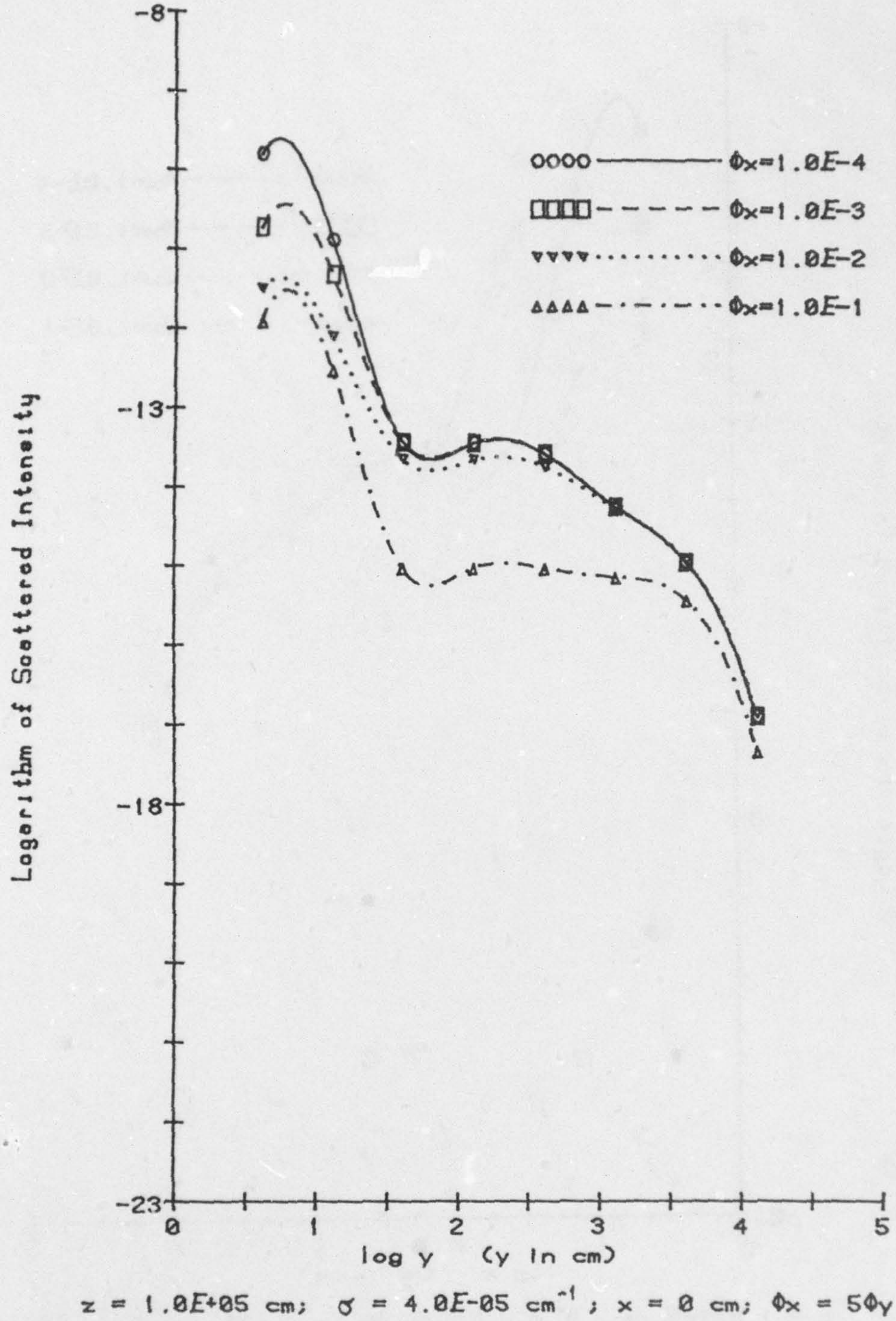

FIGURE 5e - Scattered intensity versus lateral distance y for $z = 5 \times 10^4$ cm, $\sigma = 4 \times 10^{-5}$ cm $^{-1}$, $x = 0$, $\phi_x = \phi_y$

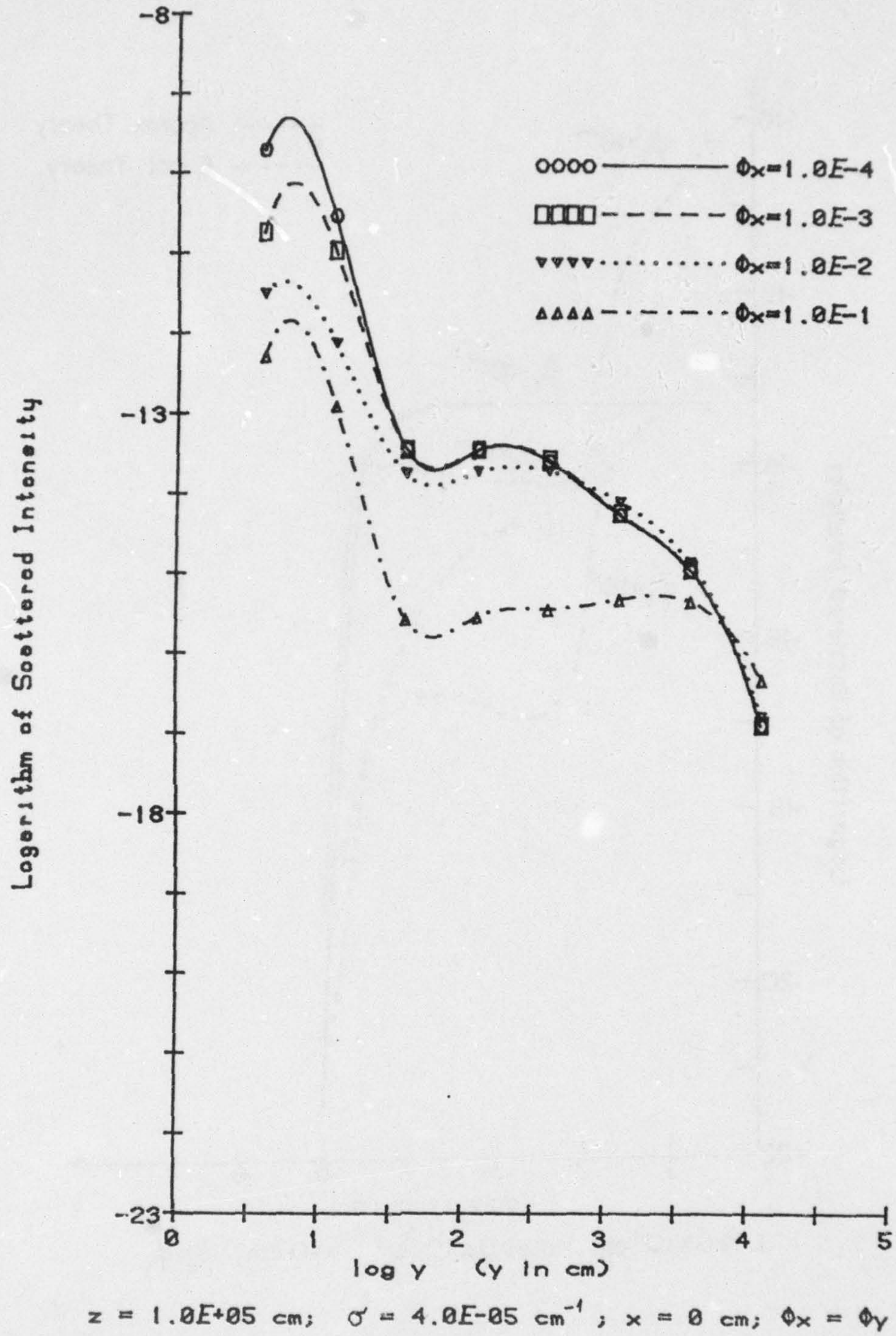

FIGURE 5f - Scattered intensity versus lateral distance x for z = 5×10^4 cm, $\sigma = 4 \times 10^{-5} cm^{-1}$, y = 0 $\phi_x = 5\phi_y$

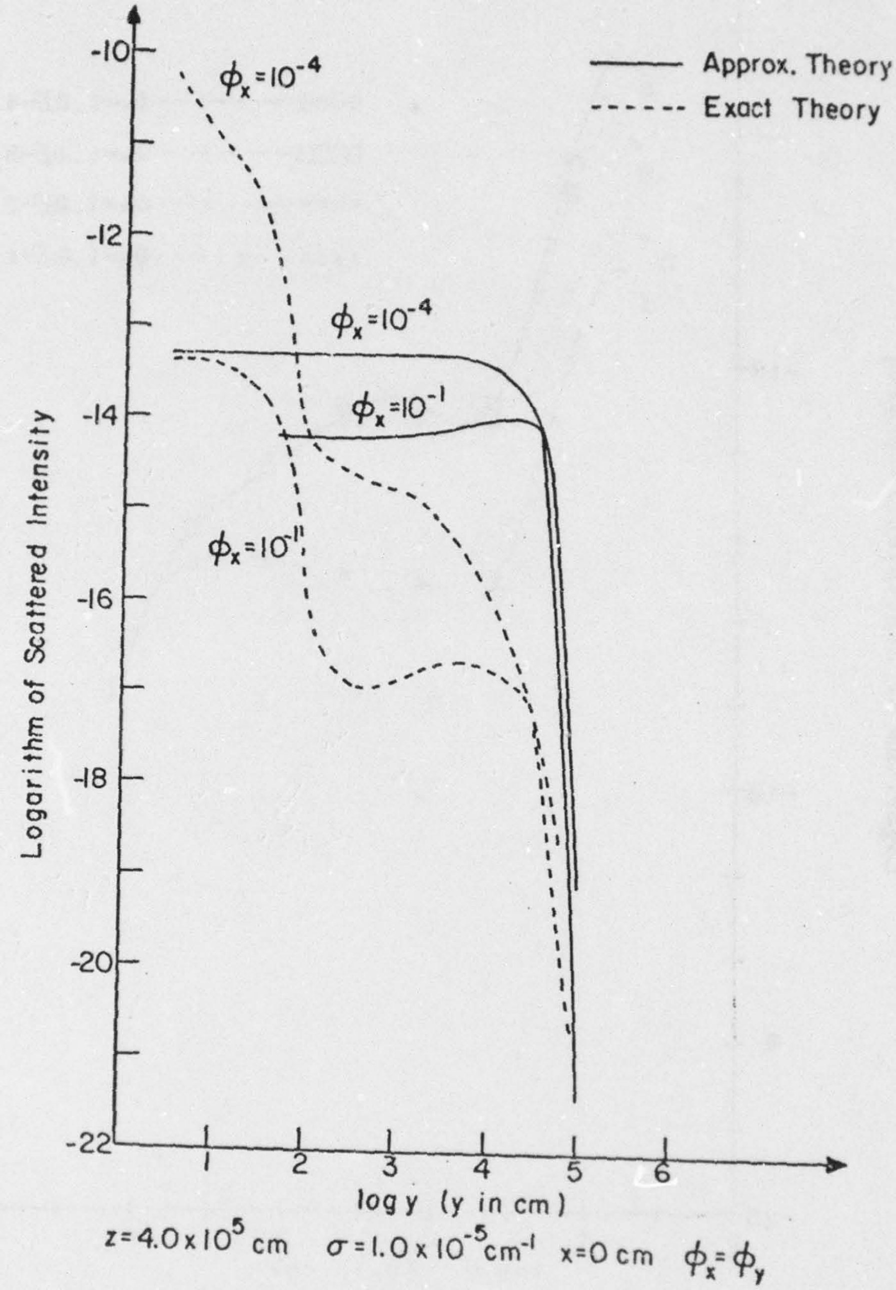

FIGURE 6 - Scattered intensity versus lateral distance y for $z = 5 \times 10^4$ cm, $\sigma = 10^{-5}$ cm $^{-1}$, $x = 0$, $\phi_x = \phi_y$ according to the exact theory and an approximate one.

4.0 CONCLUSIONS

We have derived a theoretical expression from which one can compute exactly the intensity of a laser beam undergoing small-angle multiple scattering. This theoretical expression reveals details whose presence cannot be obtained from approximate approach. In particular, we have shown the existence of three, fairly well defined, regions around the axis of the incident beam. In the first relatively narrow cylindrical region, the intensity of the scattered light is significant. Then, in the adjacent second region, the intensity drops drastically. The second region is followed by a third one where the contributions from multiple scattering become prominent.

Application of the theory discussed above to specific types of fog is in progress and the results will be presented in future reports.

5.0 REFERENCES

1. Arnush, D. "J. Opt. Soc. Am." Vol. 62, p. 9, 1972

2. Stotts, L.B. "J. Opt. Soc. Am." Vol. 67, p. 815, 1977

3. Wentzel, G. "Ann Physik" Vol. 69, p. 335, 1922

4. Ghandrasekhar, S. "Radiative Transfer", Dover, New York, 1960

5. Scott, W.S. "Rev. Mod. Phys." Vol. 35, p. 231, 1963

6. Roman, P. "Advanced Quantum Theory" Addison Wesley, New York, 1965

7. Davis, P.J. and Rabinovitch, P. "Numerical Integration" Blaisdel, Toronto, p. 141, 1967

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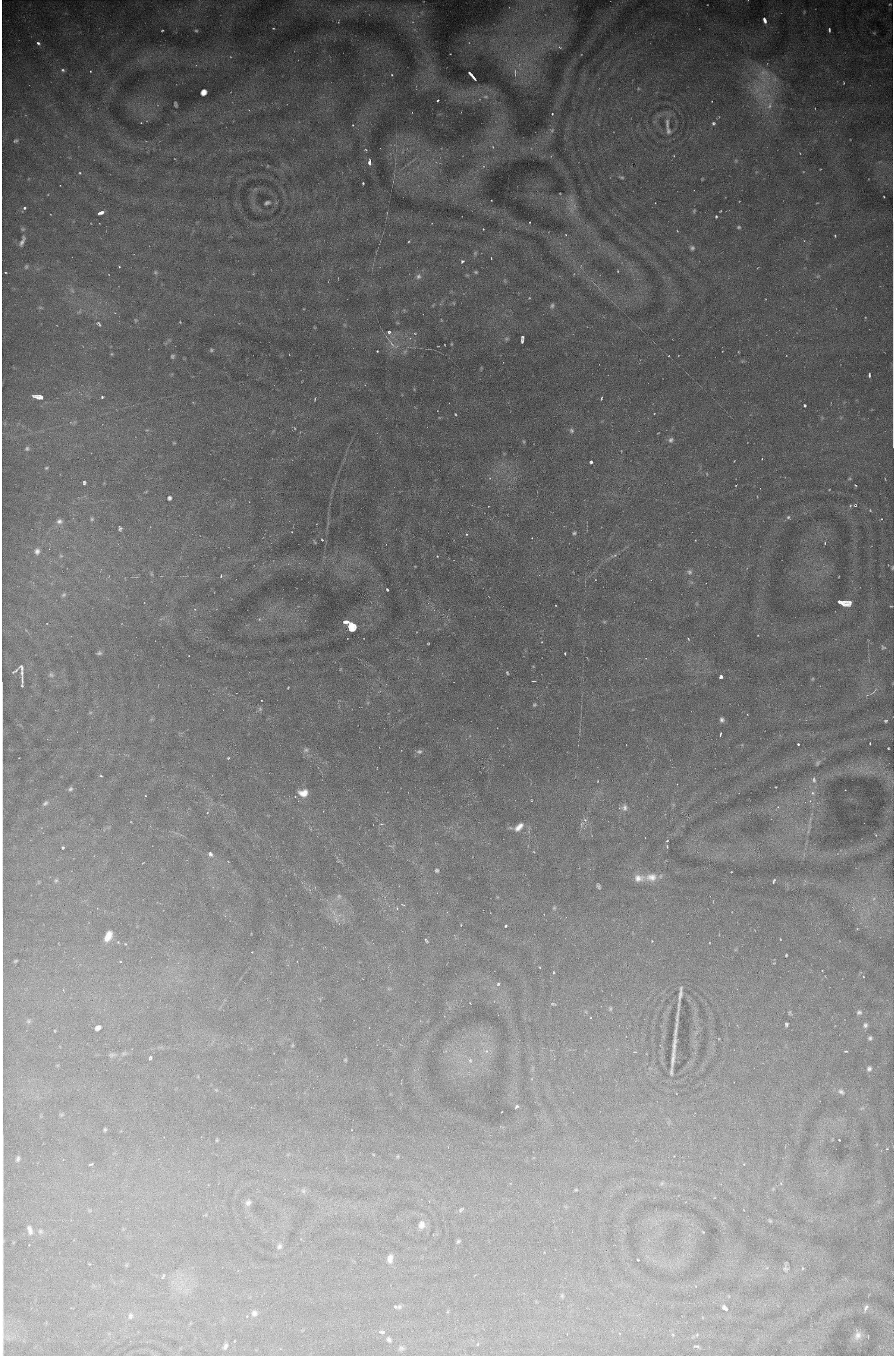
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